Rational interpolation and quadrature on the interval and on the unit circle

Karl Deckers
Adhemar Bultheel

Report TW 574, August 2010

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Rational interpolation and quadrature on the interval and on the unit circle

Karl Deckers
Adhemar Bultheel

Report TW 574, August 2010

Department of Computer Science, K.U.Leuven

Abstract

Given a positive bounded Borel measure μ on the interval $[-1, 1]$, we provide convergence results in L_2^μ-norm to a function f of its sequence of rational interpolating functions at the nodes of rational Gauss-type quadrature formulas associated with the measure μ. As an application, we construct rational interpolatory quadrature formulas for complex bounded measures σ on the interval, and give conditions to ensure the convergence of these quadrature rules. Further, an upper bound for the error on the nth approximation and an estimate for the rate of convergence is provided for these quadrature rules. Additionally, we briefly give similar results for certain rational interpolatory quadrature formulas associated with measures supported on the complex unit circle.

Keywords : Orthogonal rational functions, rational interpolation, rational quadrature rules, error bound, convergence rate.

MSC : Primary : 42C05, Secondary : 65D30, 65D32.

RATIONAL INTERPOLATION AND QUADRATURE ON THE INTERVAL AND ON THE UNIT CIRCLE*

KARL DECKERS† AND ADHEMAR BULTHEEL‡

Abstract. Given a positive bounded Borel measure μ on the interval $[-1,1]$, we provide convergence results in L_2^μ-norm to a function f of its sequence of rational interpolating functions at the nodes of rational Gauss-type quadrature formulas associated with the measure μ. As an application, we construct rational interpolatory quadrature formulas for complex bounded measures σ on the interval, and give conditions to ensure the convergence of these quadrature rules. Further, an upper bound for the error on the nth approximation and an estimate for the rate of convergence is provided for these quadrature rules. Additionally, we briefly give similar results for certain rational interpolatory quadrature formulas associated with measures supported on the complex unit circle.

Key words. Orthogonal rational functions, rational interpolation, rational quadrature rules, error bound, convergence rate.

AMS subject classifications. 42C05, 65D30, 65D32

1. Introduction. The central object of study in this paper is an integral of the form

$$J_\sigma(f) = \int_{-1}^{1} f(x) d\sigma(x),$$

where σ is a (possibly complex) bounded measure with infinite support on the interval $I = [-1,1]$. Such integrals can be approximated by interpolatory quadrature rules with interpolation points that are zeros of orthogonal polynomials and are all in I. However, since σ need not be positive, one needs to introduce an auxiliary positive orthogonality measure μ. This leads to Gauss quadrature formulas with positive weights that approximate integrals of the form $J_\mu(f)$, and have a maximal (polynomial) domain of validity. If one of the endpoints (Radau) or both of them (Lobatto) are imposed as additional nodes, we get more general Gauss-type quadrature rules. In the ideal situation μ "resembles" σ as much as possible.

However, when f has singularities outside (but possibly close to) the interval I, it is often more appropriate to not consider a maximal polynomial domain of validity, but rather consider more general spaces of rational functions. In such a case the orthogonal polynomials are replaced by orthogonal rational functions with preassigned poles (to simulate the singularities of f).

A theory of orthogonal rational functions on the complex unit circle $\mathbb{T}$ has been studied intensively (see e.g. [7]) in many papers devoted to their applications in numerical quadrature. Of course by a Joukowski Transform $x = J(z)$ one may map $x \in I$ to $z \in \mathbb{T}$ (see e.g. [3]), hence relating poles, nodes, weights, and measures on I and $\mathbb{T}$. In the classical situation, the poles for the circle situation are often taken in pairs $\{\beta_i, 1/\overline{\beta}_i\}$ with $|\beta_i| < 1$. This corresponds to taking real poles for the interval. We refer to this as the situation of "real poles"; see [25]. If, however, we want to consider arbitrary complex poles for I, then we need pairs $\{\beta_i, 1/\beta_i\}$ on $\mathbb{T}$; see [17].

*The work is partially supported by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors.

†Department of Computer Science, Katholieke Universiteit Leuven, Heverlee, Belgium. This author is a Postdoctoral Fellow of the Research Foundation - Flanders (FWO).

‡Department of Computer Science, Katholieke Universiteit Leuven, Heverlee, Belgium.

The outline of the paper is as follows. After giving the necessary theoretical background in Section 2, in Section 3 we are mainly concerned with the extension of a known L_2 convergence result with respect to a positive bounded Borel measure on $\mathbb{T}$ for sequences of interpolating rational functions with poles among $\{\beta_i, 1/\overline{\beta}_i\}$, to the case of interpolating rational functions with poles among $\{\beta_i, 1/\beta_i\}$. Once the existing result is adapted to this slightly modified situation, we are ready to study the convergence of the rational approximants to f in Section 4. The convergence is studied in L_2^μ-norm, where μ is the positive orthogonality measure on I, and the approximants interpolate f in the nodes of the corresponding rational Gauss-type quadrature rules. Then, in Section 5 we construct rational interpolatory quadrature rules to approximate the integrals of the form $J_\sigma(f)$, as well as for the approximation of integrals of the form $I_{\mathring{\sigma}}(\mathring{f}) = \int_{-\pi}^{\pi} \mathring{f}(e^{\mathbf{i}\theta}) d\mathring{\sigma}(\theta)$, where $\mathring{\sigma}$ is a (possibly complex) bounded measure with infinite support on $\mathbb{T}$. The convergence result obtained in Section 4 will of course immediately induce convergence results for the rational interpolatory quadrature formulas themselves, both on I and on $\mathbb{T}$. Next, in Section 6 we provide error bounds and an estimate for the rate of convergence (root asymptotics of the error) for these quadrature rules. We conclude with some numerical experiments in Section 7.

Similar results in the polynomial case and for the real poles situation were studied in respectively [12] and [8, 9].

2. Preliminaries. The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$ and for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, the positive half line will be represented by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $a \in \mathbb{C}$, then $\Re\{a\}$ refers to the real part of a, while $\Im\{a\}$ refers to the imaginary part, and the imaginary unit will be denoted by $\mathbf{i}$. The unit circle and the open unit disk are denoted respectively by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Whenever the value zero is omitted in a set $X \subseteq \overline{\mathbb{C}}$, this will be represented by X_0. Similarly, the complement of a set $Y \subset \overline{\mathbb{C}}$ with respect to a set $X \subseteq \overline{\mathbb{C}}$ will be denoted by X_Y; i.e., $X_Y = \{t \in X : t \notin Y\}$. Further, if $b = \lceil a \rceil$ with $a \in \mathbb{R}$, then b is the smallest integer so that $b \geq a$. If, on the other hand, $b = \lfloor a \rfloor$ with $a \in \mathbb{R}$, then b is the largest integer so that $b \leq a$.

In this paper, we will consider quadrature formulas on the interval $I = [-1, 1]$ and on the complex unit circle $\mathbb{T}$. Although x and z are both complex variables, we reserve the notation x for the interval and z for the unit circle.

For any complex function $f(t)$, with $t = z$ or $t = x$, we define the involution operation or substar conjugate by $f_*(t) = \overline{f(1/\overline{t})}$. Next, we define the super-c conjugate by $f^c(t) = \overline{f(\overline{t})}$, and consequently f_*^c by $f_*^c(t) = f(1/t)$. Note that, if $f(t)$ has a pole at $t = p$, then $f_*(t)$ (respectively $f^c(t)$ and $f_*^c(t)$) has a pole at $t = 1/\overline{p}$ (respectively $t = \overline{p}$ and $t = 1/p$). Further, with f^{inv} we denote the inverse of the function f, to avoid confusion with the notation $f^{-1} = 1/f$.

Let there be fixed a sequence of poles $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$, where the poles are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates. We then define the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = \frac{x^k}{\prod_{j=1}^k (1 - x/\alpha_j)}, \quad k = 1, 2, \ldots . \tag{2.1}$$

These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}$ defined by $\mathcal{L}_{-1} = \{0\}$, $\mathcal{L}_0 = \mathbb{C}$ and $\mathcal{L}_k := \mathcal{L}\{\alpha_1, \ldots, \alpha_k\} = \operatorname{span}\{b_0, \ldots, b_k\}$, $k = 1, 2, \ldots$. Further, with $\mathcal{L}$ we denote the closed linear span of all $\{b_k\}_{k=0}^\infty$. With the definition of the super-c conjugate we introduce $\mathcal{L}_k^c = \{f : f^c \in \mathcal{L}_k\}$. In the remainder we will also use the notation $\mathcal{L}_0^{[\alpha]} := \mathcal{L}_0$ and $\mathcal{L}_k^{[\alpha]} := \mathcal{L}\{\alpha_1, \dots, \alpha_{k-1}, \alpha\}$, $k > 0$; i.e.; the space of rational functions with the same dimension as $\mathcal{L}_k$, but with the last pole α_k replaced by α. Note that $\mathcal{L}_k$ and $\mathcal{L}_k^c$ are rational generalizations of the space $\mathcal{P}_k$ of polynomials of degree less than or equal to k. Indeed, if $\alpha_j = \infty$ for every $j \geqslant 1$, the expression in (2.1) becomes $b_k(x) = x^k$.

Consider the integral

$$J_\sigma(f) := \int_{-1}^{1} f(x) d\sigma(x),$$

where σ is a (possibly complex) bounded measure with infinite support on I (in short, a complex measure on I). To approximate $J_\sigma(f)$, where f is a possibly complex function that can have singularities (possibly close to, but) outside the interval, rational interpolatory quadrature formulas are often preferred. An nth rational interpolatory quadrature is obtained by integrating an interpolating rational function of degree $n-1$, and is of the form

$$J_n^\sigma(f) := \sum_{k=1}^{n} \lambda_{n,k}^\sigma f(x_{n,k}), \quad \{x_{n,k}\}_{k=1}^n \subset I, \quad x_{n,j} \neq x_{n,k} \text{ if } j \neq k, \quad \{\lambda_{n,k}^\sigma\}_{k=1}^n \subset \mathbb{C},$$

so that $J_\sigma(f) = J_n^\sigma(f)$ for every $f \in \mathcal{R}_{p,q} = \mathcal{L}_p \cdot \mathcal{L}_q^c$, with $p+q \leqslant 2n-1$ and $p, q \leqslant n$. For reasons of notational simplicity, in the remainder we will write x_k and λ_k^σ, meaning $x_{m,k}$ and $\lambda_{m,k}^\sigma$ for a certain index m. At any time, the index m should be clear from the context.

Next, consider the inner product defined by

$$\langle f, g \rangle_\mu = J_\mu(f g^c), \quad f, g \in \mathcal{L}, \tag{2.2}$$

where μ is a positive bounded Borel measure with infinite support on I (in short, a positive measure on I), and let $\|f\|_{\mu,2} := \sqrt{\langle f, f \rangle_\mu}$. Orthogonalizing the basis functions $\{b_0, b_1, \dots\}$ with respect to this inner product, we obtain a sequence of orthogonal rational functions (ORFs) $\{\varphi_0, \varphi_1, \dots\}$, with $\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, so that $\varphi_k \perp_\mu \mathcal{L}_{k-1}$; i.e.; $\langle \varphi_k, \varphi_j \rangle_\mu = d_k \delta_{k,j}$, $d_k \in \mathbb{R}_0^+$ and $k, j = 0, 1, \dots$, where $\delta_{k,j}$ is the Kronecker Delta.

Let $\alpha \in \overline{\mathbb{R}}_I$, and suppose $\varphi_n^{[\alpha]} \in \mathcal{L}_n^{[\alpha]} \setminus \mathcal{L}_{n-1}$ is orthogonal to $\mathcal{L}_{n-1}$ with respect to the inner product (2.2). Then the zeros x_k of $\varphi_n^{[\alpha]}(x)$ are all distinct and in the open interval $(-1, 1)$, and hence, can be chosen as nodes for the quadrature formula $J_n^\sigma(f)$. For $\sigma = \mu$ and $\alpha_n = \alpha$, we obtain in this way the n-point rational Gaussian quadrature formula, which has maximal domain of validity; i.e.; the approximation is exact for every function $f \in \mathcal{R}_{n,n-1}$. It is well known that the weights λ_k^μ in the rational Gaussian quadrature are all positive (see e.g. [14, Thm. 2.3.5]). Note, however, that the n-point rational Gaussian quadrature formula does not exist whenever the last pole $\alpha_n \notin \overline{\mathbb{R}}$. If $\alpha_n \notin \overline{\mathbb{R}}$ or $\alpha_n \neq \alpha$, the n-point rational quadrature formula based on the zeros of the ORF $\varphi_n^{[\alpha]}$ becomes an n-point rational Gauss-Radau quadrature formula with positive weights (see below).

For any other choice of nodes, the weights may be non-positive or even complex and the quadrature will only be exact in a smaller set of rational functions. For each node that is fixed in advance, the domain of validity will generally[1] decrease by one.

[1]For some specific choices for the nodes, the domain of validity may remain the same or may even decrease more (see also [2] for the polynomial case).

It is reasonable though to require that we have at least exactness for $p+q=n-1$. A special case is obtained when one node in the n-point quadrature is fixed in advance, so that the weights are all positive and the quadrature is exact for every $f \in \mathcal{R}_{n-1,n-1}$, which corresponds to the n-point rational Gauss-Radau quadrature formula. However, the existence of this n-point rational Gauss-Radau quadrature depends on the choice of the node (e.g., it surely does not exist whenever the node is a zero of the ORF φ_{n-1}; see [16]), but it does not depend on whether $\alpha_n \in \overline{\mathbb{R}}_I$. Whenever two nodes in an $(n+1)$-point quadrature formula are fixed in advance, so that the weights are all positive and the quadrature is exact for every $f \in \mathcal{R}_{n,n-1}$, we obtain the $(n+1)$-point rational Gauss-Lobatto quadrature formula. The existence of this $(n+1)$-point rational Gauss-Lobatto quadrature not only depends now on the choice of the nodes, but also on the pole α_n (e.g., it is easily verified that it does not exist whenever $\alpha_n \notin \overline{\mathbb{R}}$; see also [15, Sect. 2]).

Another sequence of basis functions will be used for the unit circle. Given a sequence of complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, we define the Blaschke products for $\mathcal{B}$ as

$$B_0(z) \equiv 1, \quad B_k(z) = \prod_{j=1}^{k} \frac{z-\beta_j}{1-\overline{\beta}_j z}, \ k = 1, 2, \ldots . \tag{2.3}$$

These Blaschke products generate the nested spaces of rational functions $\mathring{\mathcal{L}}_{-1} = \{0\}$, $\mathring{\mathcal{L}}_0 = \mathbb{C}$ and $\mathring{\mathcal{L}}_k := \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_k\} = \operatorname{span}\{B_0, \ldots, B_k\}$, $k = 1, 2, \ldots$. Similarly as before, we denote with $\mathring{\mathcal{L}}$ the closed linear span of all $\{B_k\}_{k=0}^{\infty}$. With the definition of the substar conjugate and the super-c conjugate we can define $\mathring{\mathcal{L}}_{k*} = \{f : f_* \in \mathring{\mathcal{L}}_k\}$, $\mathring{\mathcal{L}}_k^c = \{f : f^c \in \mathring{\mathcal{L}}_k\}$ and $\mathring{\mathcal{L}}_{k*}^c = \{f : f_*^c \in \mathring{\mathcal{L}}_k\}$. Also here we will use the notation $\mathring{\mathcal{L}}_0^{[\beta]} := \mathring{\mathcal{L}}_0$ and $\mathring{\mathcal{L}}_k^{[\beta]} := \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_{k-1}, \beta\}$, $k > 0$, to denote the space of rational functions with the same dimension as $\mathring{\mathcal{L}}_k$, but with the last complex number β_k replaced by β. Note that $\mathring{\mathcal{L}}_k$ and $\mathring{\mathcal{L}}_k^c$ are rational generalizations of $\mathcal{P}_k$ too. Indeed, if all $\beta_j = 0$ (or equivalently, $1/\overline{\beta}_j = \infty$ for every $j \geqslant 1$), the expression in (2.3) becomes $B_k(z) = B_k^c(z) = z^k$.

Consider now the integral

$$I_{\mathring{\sigma}}(f) := \int_{-\pi}^{\pi} f(z) d\mathring{\sigma}(\theta), \ z = e^{\mathbf{i}\theta},$$

where $\mathring{\sigma}$ is a complex measure on $\mathbb{T}$[2], and f is a (possibly complex) function bounded on $\mathbb{T}$. The rational interpolatory quadrature formulas to approximate $I_{\mathring{\sigma}}(f)$ are then of the form

$$I_n^{\mathring{\sigma}}(f) := \sum_{k=1}^{n} \mathring{\lambda}_{n,k}^{\mathring{\sigma}} f(z_{n,k}), \quad \{z_{n,k}\}_{k=1}^n \subset \mathbb{T}, \quad z_{n,j} \neq z_{n,k} \text{ if } j \neq k, \quad \{\mathring{\lambda}_{n,k}^{\mathring{\sigma}}\}_{k=1}^n \subset \mathbb{C},$$

so that $I_{\mathring{\sigma}}(f) = I_n^{\mathring{\sigma}}(f)$ for every $f \in \mathring{\mathcal{R}}_{p,q} = \mathring{\mathcal{L}}_p \cdot \mathring{\mathcal{L}}_{q*}$, with $n-1 \leqslant p+q \leqslant 2n-2$ and $p, q \leqslant n-1$. From now on we will write z_k and $\mathring{\lambda}_k^{\mathring{\sigma}}$, meaning $z_{m,k}$ and $\mathring{\lambda}_{m,k}^{\mathring{\sigma}}$ for a certain index m, where the index m should again be clear at any time from the context.

Let $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$ denote an nth ORF with respect to the inner product

$$\langle f, g \rangle_{\mathring{\mu}} = I_{\mathring{\mu}}(f g_*), \quad f, g \in \mathring{\mathcal{L}},$$

[2] The measure $\mathring{\sigma}$ on $\mathbb{T}$ induces a measure on $[-\pi, \pi]$ for which we shall use the same notation $\mathring{\sigma}$.

where $\mathring{\mu}$ is a positive measure on $\mathbb{T}$, and let $\|f\|_{\mathring{\mu},2} := \sqrt{\langle f\,,f\rangle_{\mathring{\mu}}}$. We then define a para-orthogonal rational function

$$\mathring{Q}_{n,\tau}(z) = \phi_n(z) + \tau B_n(z)\phi_{n*}(z), \quad \tau \in \mathbb{T}. \tag{2.4}$$

The zeros z_k of $\mathring{Q}_{n,\tau}(z)$ are all distinct and on the unit circle $\mathbb{T}$, and hence, can be chosen as nodes for the quadrature formula $I_n^{\mathring{\sigma}}(f)$. In the special case in which $\mathring{\sigma} = \mathring{\mu}$, we obtain an n-point rational Szegő quadrature formula, which has maximal domain of validity ($p = q = n-1$). It is well known that in this case the weights $\mathring{\lambda}_k^{\mathring{\mu}}$ are all positive too.

Due to the presence of the parameter τ in (2.4), the nodes and weights in an n-point rational Szegő quadrature formula are (unlike in the case of the interval) not unique. Consequently, an n-point rational Szegő-Radau quadrature formula (one fixed node) with positive weights always exists and has maximal domain of validity too. While an n-point rational Szegő-Lobatto quadrature formula (two fixed nodes) with positive weights is at least exact for every $f \in \mathring{\mathcal{R}}_{n-2,n-2}$ and again always exists (see e.g. [5] and [11]).

We denote the Joukowski Transformation $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}_I$ and the unit circle $\mathbb{T}$ onto the interval I. When $z = e^{\mathbf{i}\theta}$, then $x = J(z) = \cos\theta$. In this paper we will assume that x and z are related by this transformation. The inverse mapping is denoted by $z = J^{inv}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}_I$. With the sequence $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ we associate a sequence $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, so that $\beta_k = J^{inv}(\alpha_k)$ for every $k > 0$, and $\hat{\mathcal{B}} = \{\hat{\beta}_1, \hat{\beta}_2, \ldots\} \subset \mathbb{D}$ with $\hat{\beta}_{2k} = \overline{\hat{\beta}}_{2k-1} = \beta_k$, $k = 1, 2, \ldots$. Further, we denote the nested spaces of rational functions based on the sequence $\hat{\mathcal{B}}$ by $\hat{\mathring{\mathcal{L}}}_k := \mathring{\mathcal{L}}\{\hat{\beta}_1, \ldots, \hat{\beta}_k\}$, so that

$$\hat{\mathring{\mathcal{L}}}_{2k} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_k \quad \text{and} \quad \hat{\mathring{\mathcal{L}}}_{2k-1} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k-1}. \tag{2.5}$$

A connection between quadrature formulas on the unit circle and the interval I is given in e.g. [3] and [4]. If σ is a complex measure on I, we obtain a complex measure on $\mathbb{T}$ by setting

$$\mathring{\sigma}(E) = \sigma\left(\{\cos\theta, \theta \in E \cap [0,\pi)\}\right) + \sigma\left(\{\cos\theta, \theta \in E \cap [-\pi,0)\}\right). \tag{2.6}$$

Clearly, this measure $\mathring{\sigma}$ is then symmetric (i.e.; $d\mathring{\sigma}(-\theta) = -d\mathring{\sigma}(\theta)$), so that $I_{\mathring{\sigma}}(f_*^c) = I_{\mathring{\sigma}}(f)$ for every function f on $\mathbb{T}$.

Note that by the Joukowski Transformation, a function $f(x)$ transforms into a function $\mathring{f}(z) = (f \circ J)(z)$, so that $\mathring{f}_*^c(z) = \mathring{f}(z)$ and

$$J_\sigma(f) = \frac{1}{2} I_{\mathring{\sigma}}(\mathring{f}). \tag{2.7}$$

Further, let $\mathring{\mathcal{S}}_n^{[c]}$ and $\mathring{\mathcal{S}}_n^{[*]}$ be defined by

$$\mathring{\mathcal{S}}_{2k-1}^{[c]} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{(k-1)*}\,, \quad \mathring{\mathcal{S}}_{2k-1}^{[*]} = \mathring{\mathcal{L}}_{k-1}^c \cdot \mathring{\mathcal{L}}_{k*} \;\left(= \mathring{\mathcal{S}}_{(2k-1)*}^{[c]\;c}\right),$$
$$\text{and} \quad \mathring{\mathcal{S}}_{2k}^{[c]} = \mathring{\mathcal{S}}_{2k}^{[*]} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k*} = \mathring{\mathcal{S}}_{2k},$$

and let $\mathring{\mathcal{S}} = \mathring{\mathcal{L}}^c \cdot \mathring{\mathcal{L}}_* = \mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$. From [18, Lem. 3.1] it then follows that every function $f \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a function $\mathring{f} \in \mathring{\mathcal{S}}_{2k} \setminus \left(\mathring{\mathcal{S}}_{2k-1}^{[c]} \cup \mathring{\mathcal{S}}_{2k-1}^{[*]}\right)$; see also [14, Chapt. 3.2].

Consider an arbitrary set of m distinct nodes $\mathbf{x_m} := \{x_k\}_{k=1}^m \subset (-1,1)$, and let $\mathbf{z_{2m}} := \{z_k\}_{k=1}^{2m} \subset \mathbb{T}\setminus\{-1,1\}$ be the corresponding $2m$ distinct nodes on the unit circle, with $z_k = \overline{z}_{m+k}$ and $x_k = J(z_k) = J(z_{m+k})$ for $k = 1,\dots,m$. We then distinguish the following three cases:

(0) $\mathbf{x_n^{[0]}} = \mathbf{x_n}$, and hence, $\mathbf{z_{2n}^{[0]}} = \mathbf{z_{2n}}$;

(1) $\mathbf{x_n^{[1]}} = \mathbf{x_{n-1}} \cup \{\pm 1\}$, and hence, $\mathbf{z_{2n-1}^{[1]}} = \mathbf{z_{2n-2}} \cup \{\pm 1\}$;

(2) $\mathbf{x_{n+1}^{[2]}} = \mathbf{x_{n-1}} \cup \{1,-1\}$, and hence, $\mathbf{z_{2n}^{[2]}} = \mathbf{z_{2n-2}} \cup \{1,-1\}$.

Let the parameter $s \in \{0,1,2\}$ refer to one of this three cases, and let $n(s)$ and $N(s)$ be defined by

$$n(s) = n + \lfloor s/2 \rfloor \qquad \text{and} \qquad N(s) = 2n - s \bmod 2.$$

This way we can briefly write that the set of $n(s)$ distinct nodes $\mathbf{x_{n(s)}^{[s]}} \subset I$ corresponds with the set of $N(s)$ distinct nodes $\mathbf{z_{N(s)}^{[s]}} \subset \mathbb{T}$, where s represents the number of nodes that are equal to 1 in absolute value. We then have the following theorem, which has been proved in [15, Sect. 3].

Theorem 1. *Let $\alpha = J(\beta) \in \overline{\mathbb{R}}_I$, and suppose the positive measures μ on I and $\mathring{\mu}$ on $\mathbb{T}$ are related by (2.6). Then the rational interpolatory quadrature $J_\mu(f) \approx J^\mu_{n(s)}(f)$ based on the nodes $\mathbf{x_{n(s)}^{[s]}} \subset I$ is exact for every*

$$f \in \begin{cases} \mathcal{R}_{n-1,n-1}, & s = 1 \\ \mathcal{L}_n^{[\alpha]} \cdot \mathcal{L}_{n-1}^c, & s = 0 \text{ or } s = 2. \end{cases}$$

iff the rational interpolatory quadrature $I_{\mathring{\mu}}(\mathring{f}) \approx I^{\mathring{\mu}}_{N(s)}(\mathring{f})$ based on the nodes $\mathbf{z_{N(s)}^{[s]}} \subset \mathbb{T}$ is exact for every

$$\mathring{f} \in \begin{cases} \mathring{\hat{\mathcal{L}}}_{2n-2} \cdot \mathring{\hat{\mathcal{L}}}_{(2n-2)*}, & s = 1 \\ \mathring{\hat{\mathcal{L}}}^{[\beta]}_{2n-1} \cdot \mathring{\hat{\mathcal{L}}}^{[\beta]}_{(2n-1)*}, & s = 0 \text{ or } s = 2. \end{cases}$$

In other words, certain rational Gauss-type (Gaussian, Gauss-Radau and Gauss-Lobatto) quadratures on I are related with certain rational Szegő quadratures on $\mathbb{T}$, and vice versa. Moreover, for these specific rational Gauss-type quadratures it holds that $J^\mu_{n(s)}(f) = \frac{1}{2} I^{\mathring{\mu}}_{N(s)}(\mathring{f})$ for every function $f(x)$ on I, with corresponding function $\mathring{f}(z) = (f \circ J)(z)$ on $\mathbb{T}$.

In the remainder we will refer to the nodes of the specific rational Gauss-type quadratures in Theorem 1 as (μ, s, α)-nodes or (μ, s)-nodes. The nodes in the corresponding rational Szegő quadratures in Theorem 1 will then be referred to as $(\mathring{\mu}, s, \beta)$-nodes or $(\mathring{\mu}, s)$-nodes. Note that Theorem 1 also deals with a special kind of rational Gauss-Radau quadratures with nodes inside $(-1,1)$; i.e.; when $\alpha \neq \alpha_n$. However, depending on the measure μ, this does not necessary cover all the possible kind of rational Gauss-Radau quadratures with nodes inside $(-1,1)$. Neither does Theorem 1 covers all the possible rational Szegő quadratures, because it only considers sequences of nodes on $\mathbb{T}$ where the nodes, different from 1 and -1, appear in complex conjugate pairs. For this reason, we will also use the term '$\mathring{\mu}$-nodes', meaning that we consider the more general situation of an arbitrary sequence of rational Szegő–nodes on $\mathbb{T}$ associated with a positive measure $\mathring{\mu}$ that is not necessarily symmetric.

3. The space of rational functions $\mathring{\mathcal{S}}$. In [6, 8, 9] the $L_2^{\mathring{\mu}}(\mathbb{T})$ convergence has been studied for sequences of interpolating rational functions in nested subspaces of $\mathring{\mathcal{R}} = \mathring{\mathcal{L}} + \mathring{\mathcal{L}}_*$ to a function $\mathring{f}(z)$ defined on $\mathbb{T}$. The subspaces under consideration were of the form $\mathring{\mathcal{R}}_{p(n),q(n)}$, with $p(n) + q(n) = n$, and

$$\lim_{n\to\infty} \frac{q(n)}{n} = r \in (0,1), \tag{3.1}$$

so that $\mathring{\mathcal{R}}_{p(n-1),q(n-1)} \subset \mathring{\mathcal{R}}_{p(n),q(n)}$ for every $n > 0$. In the next section we will study the $L_2^{\mu}(I)$ convergence when considering sequences of interpolating rational functions of increasing degree to a function $f(x)$ defined on I. This will be done by passing from the interval to the unit circle by means of the Joukowski Transformation $x = J(z)$. However, from the previous section we deduce that the corresponding interpolating rational functions for the function $\mathring{f}(z) = (f \circ J)(z)$ on $\mathbb{T}$ are in nested subspaces of $\mathring{\mathcal{S}} = \mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$, and of the form $\mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$. Hence, these subspaces differ from those considered in the references, unless the numbers β_k, $k = 1, 2, \ldots$, are all real. The aim of this section is to extend [9, Thm. 4.1], for the case of nested subspaces of $\mathring{\mathcal{R}}$, to the case of nested subspaces of $\mathring{\mathcal{S}}$. The main result can be found in Theorem 6, but first we need the following four lemmas.

Lemma 2. *Let the rational spaces $\mathring{\mathcal{S}}_n^{[c]}$ and $\mathring{\mathcal{S}}_n^{[*]}$ be defined as above, and define $\hat{\mathring{\mathcal{R}}}_{n,n} = \hat{\mathring{\mathcal{L}}}_n \cdot \hat{\mathring{\mathcal{L}}}_{n*}$ and $\hat{\mathring{\mathcal{R}}}^c_{n,n} = \hat{\mathring{\mathcal{L}}}^c_n \cdot \hat{\mathring{\mathcal{L}}}^c_{n*}$, where $\hat{\mathring{\mathcal{L}}}_n$ is given by (2.5). Then for every function f and g in $\mathring{\mathcal{S}}_n^{[c]}$ it holds that $fg_* \in \hat{\mathring{\mathcal{R}}}_{n,n}$. Similarly, for every function f and g in $\mathring{\mathcal{S}}_n^{[*]}$ it holds that $fg_* \in \hat{\mathring{\mathcal{R}}}^c_{n,n}$.*

Proof. We will only give the proof for the case in which f and g are in $\mathring{\mathcal{S}}_n^{[c]}$ (the case in which f and g are in $\mathring{\mathcal{S}}_n^{[*]}$ can be proved in a similar way). For $n = 2k$ we have that

$$\mathring{\mathcal{S}}_n^{[c]} \cdot \mathring{\mathcal{S}}_{n*}^{[c]} = \left(\mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_{k*}\right) \cdot \left(\mathring{\mathcal{L}}^c_{k*} \cdot \mathring{\mathcal{L}}_k\right) = \left(\mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_k\right) \cdot \left(\mathring{\mathcal{L}}^c_{k*} \cdot \mathring{\mathcal{L}}_{k*}\right) = \hat{\mathring{\mathcal{L}}}_n \cdot \hat{\mathring{\mathcal{L}}}_{n*} = \hat{\mathring{\mathcal{R}}}_{n,n}.$$

On the other hand, for $n = 2k - 1$ we have that

$$\begin{aligned}\mathring{\mathcal{S}}_n^{[c]} \cdot \mathring{\mathcal{S}}_{n*}^{[c]} &= \left(\mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_{(k-1)*}\right) \cdot \left(\mathring{\mathcal{L}}^c_{k*} \cdot \mathring{\mathcal{L}}_{k-1}\right) \\ &= \left(\mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_{k-1}\right) \cdot \left(\mathring{\mathcal{L}}^c_{k*} \cdot \mathring{\mathcal{L}}_{(k-1)*}\right) = \hat{\mathring{\mathcal{L}}}_n \cdot \hat{\mathring{\mathcal{L}}}_{n*} = \hat{\mathring{\mathcal{R}}}_{n,n}.\end{aligned}$$

This ends the proof. ∎

Lemma 3. *Let $\mathring{\lambda}$ denote the Lebesgue measure on $\mathbb{T}$. Then the space $\mathring{\mathcal{S}}$ is dense in $L_p^{\mathring{\lambda}}(\mathbb{T})$, $1 \leqslant p < \infty$, and in the class $C(\mathbb{T})$ of continuous 2π-periodic functions on $\mathbb{T}$ (with respect to the Chebyshev norm) iff $\sum_{k=1}^{\infty}(1 - |\beta_k|) = \infty$.*

Proof. The statement has been proved in [7, Thm. 7.1.2] for the space $\mathring{\mathcal{R}} = \mathring{\mathcal{L}} + \mathring{\mathcal{L}}_*$ with poles among $\{\beta_k, 1/\overline{\beta}_k\}_{k=0}^{\infty}$. It is easily verified that the proof in [7] remains valid when considering the space $\mathring{\mathcal{S}} = \mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$ with poles among $\{\beta_k, 1/\beta_k\}_{k=0}^{\infty}$ instead. ∎

Lemma 4. *Let the symbol 'ι' be fixed to either 'c' or to '$*$'. Suppose $\mathring{\mu}$ is a positive measure on $\mathbb{T}$, and consider the nested spaces of rational functions $\mathring{\mathcal{S}}_n^{[\iota]}$. Then*

1. *For each $n = 1, 2, \ldots$, there exist n distinct points $\{z_k\}_{k=1}^n$ on $\mathbb{T}$ and n positive numbers $\mathring{\lambda}_k$, so that $\tilde{I}_n(\mathring{f}) := \sum_{k=1}^n \mathring{\lambda}_k \mathring{f}(z_k) = I_{\mathring{\mu}}(\mathring{f})$ for every function $\mathring{f}$ of the form $\mathring{f} = fg_*$, with $f, g \in \mathring{\mathcal{S}}_{n-1}^{[\iota]}$.*

2. If $\sum_{k=1}^{\infty}(1-|\beta_k|^2)=\infty$, then $\lim_{n\to\infty}\tilde{I}_n(\mathring{f})=I_{\mathring{\mu}}(\mathring{f})$ for all $\mathring{\mu}$-integrable functions $\mathring{f}$.

Proof. The first statement has been proved in [10, Thm. 1] for functions $\mathring{f}$ of the form $\mathring{f}=\tilde{f}\tilde{g}_*$, with $\tilde{f},\tilde{g}\in\mathring{\mathcal{L}}_{n-1}$. Clearly, the statement remains valid for $\tilde{f},\tilde{g}\in\hat{\mathring{\mathcal{L}}}_{n-1}$ and for $\tilde{f},\tilde{g}\in\hat{\mathring{\mathcal{L}}}^c_{n-1}$. As a result of Lemma 2, we have that for every f and g in $\mathring{\mathcal{S}}^{[c]}_{n-1}$ (respectively in $\mathring{\mathcal{S}}^{[*]}_{n-1}$) there exist $\tilde{f}$ and $\tilde{g}$ in $\hat{\mathring{\mathcal{L}}}_{n-1}$ (respectively in $\hat{\mathring{\mathcal{L}}}^c_{n-1}$), so that $fg_*=\tilde{f}\tilde{g}_*$. This proves the first statement.

Next, from the proof of Lemma 2 it also follows that for every $\tilde{f}$ and $\tilde{g}$ in $\hat{\mathring{\mathcal{L}}}_{n-1}$ (respectively in $\hat{\mathring{\mathcal{L}}}^c_{n-1}$), there exist f and g in $\mathring{\mathcal{S}}^{[c]}_{n-1}$ (respectively in $\mathring{\mathcal{S}}^{[*]}_{n-1}$), so that $\tilde{f}\tilde{g}_*=fg_*$. Consequently, $\tilde{I}_n(\mathring{f})$ is a rational Szegő quadrature formula. The second statement now follows from [10, Thm. 5 and Cor. 6], and the fact that $\sum_{k=1}^{\infty}(1-\left|\hat{\beta}_k\right|^2)=\infty$ iff $\sum_{k=1}^{\infty}(1-|\beta_k|^2)=\infty$. This ends the proof. ■

Lemma 5. *Let $\mathring{\mu}$ be a positive measure on $\mathbb{T}$, and consider a function $\mathring{f}$, bounded on $\mathbb{T}$, for which the Riemann–Stieltjes integral $I_{\mathring{\mu}}(\mathring{f})$ exists. Let $\mathring{\mathcal{T}}^{[P]}_n$, $n=0,1,\ldots$, represent the classes of (complex) trigonometric functions of degree n at most, and set $\mathring{\mathcal{T}}^{[P]}=\bigcup_{n=0}^{\infty}\mathring{\mathcal{T}}^{[P]}_n$. Further, assume $\sum_{k=1}^{\infty}(1-|\beta_k|)=\infty$, and let the classes $\mathring{\mathcal{T}}^{[R]}_n$, $n=0,1,\ldots$, of (complex) rational trigonometric functions be defined by*

$$\mathring{\mathcal{T}}^{[R]}_n=\left\{R(\theta)=\frac{P(\theta)}{\prod_j^n(1-\beta_j e^{\mathbf{i}\theta})(e^{\mathbf{i}\theta}-\beta_j)}:P\in\mathring{\mathcal{T}}^{[P]}_n\right\},$$

and $\mathring{\mathcal{T}}^{[R]}=\bigcup_{n=0}^{\infty}\mathring{\mathcal{T}}^{[R]}_n$. Then for any $\epsilon>0$ there exists a rational trigonometric function $R\in\mathring{\mathcal{T}}^{[R]}$, so that

$$\left\|\mathring{f}-R\right\|^2_{\mathring{\mu},2}<2(M+\epsilon)\epsilon,$$

where

$$M=\sup_{\theta\in[-\pi,\pi]}\left|\Re\{\mathring{f}(e^{\mathbf{i}\theta})\}\right|+\sup_{\theta\in[-\pi,\pi]}\left|\Im\{\mathring{f}(e^{\mathbf{i}\theta})\}\right|. \tag{3.2}$$

Proof. Let ϵ' be an arbitrary positive number. From [23, Thm. 1.5.4] and the proof of [9, Thm. 4.1] it follows that there exists a trigonometric polynomial $P\in\mathring{\mathcal{T}}^{[P]}$, so that

$$I_{\mathring{\mu}}(|f-P|)<\epsilon' \quad\text{and}\quad |f-P|<2(M+\epsilon'),$$

where M is given by (3.2). Under the condition $\sum_{k=1}^{\infty}(1-|\beta_k|)=\infty$, it follows from Lemma 3 that $\mathring{\mathcal{T}}^{[R]}$ is dense in $C(\mathbb{T})$, while P is in $C(\mathbb{T})$. Consequently, there exist sequences $\{R_n\}$ in $\mathring{\mathcal{T}}^{[R]}$ such that

$$\lim_{n\to\infty}R_n=P,\quad\text{uniformly in }[-\pi,\pi].$$

Thus, for every $\epsilon''>0$ we can find an integer k so that for all $n\geqslant k$:

$$|P-R_n|<2\epsilon'',\quad\forall\theta\in[-\pi,\pi].$$

Because the convergence $R_n \to P$ is uniform in $[-\pi, \pi]$, we also have

$$\lim_{n\to\infty} I_{\mathring{\mu}}(R_n) = I_{\mathring{\mu}}(P).$$

Thus, for every $\epsilon'' > 0$ we can find an integer l so that for all $n \geqslant l$:

$$I_{\mathring{\mu}}\left(|P - R_n|\right) < \epsilon''.$$

Setting $R = R_N \in \mathring{\mathcal{T}}^{[R]}$ for a certain $N > \max\{k, l\}$, we have that

$$|f - R| \leqslant |f - P| + |P - R| < 2(M + \epsilon' + \epsilon''),$$

and

$$I_{\mathring{\mu}}(|f - R|) \leqslant I_{\mathring{\mu}}(|f - P|) + I_{\mathring{\mu}}(|P - R|) < \epsilon' + \epsilon''.$$

Setting $\epsilon = (\epsilon' + \epsilon'')$, we obtain in this way that

$$\left\|\mathring{f} - R\right\|^2_{\mathring{\mu},2} < 2(M + \epsilon) \cdot I_{\mathring{\mu}}(|f - R|) < 2(M + \epsilon)\epsilon.$$

This concludes the proof. ■

We are now in a position to prove the following main result on $L_2^{\mathring{\mu}}(\mathbb{T})$ convergence that will be needed in the next section to study the $L_2^{\mu}(I)$ convergence.

Theorem 6. *Let the symbol 'ι' be fixed to either 'c' or to '$*$', and assume $\sum_{j=1}^{\infty}(1 - |\beta_j|) = \infty$. Suppose $\mathring{\mu}$ is a positive measure on $\mathbb{T}$, and consider the nested spaces of rational functions $\mathring{\mathcal{S}}_n^{[\iota]}$. Further, suppose $\{z_k^{[\iota]}\}_{k=1}^n$, $n = 1, 2, \ldots$, are sets of n distinct $\mathring{\mu}$-nodes on $\mathbb{T}$. Then for any function $\mathring{f}$ bounded on $\mathbb{T}$, for which the Riemann–Stieltjes integral $I_{\mathring{\mu}}(\mathring{f})$ exists, the sequence of interpolating rational functions $\left\{S_{n-1}^{\mathring{f}\,[\iota]}\right\}_{n=1}^{\infty}$, with $S_{n-1}^{\mathring{f}\,[\iota]} \in \mathring{\mathcal{S}}_{n-1}^{[\iota]}$, at the points $\{z_k^{[\iota]}\}_{k=1}^n$ converge to $\mathring{f}$ in $L_2^{\mathring{\mu}}$-norm; i.e;*

$$\lim_{n\to\infty} \left\|\mathring{f} - S_{n-1}^{\mathring{f}\,[\iota]}\right\|_{\mathring{\mu},2} = 0.$$

Proof. The proof is the same as in [9, Thm. 4.1] when replacing Theorem 2.1, Theorem 2.2 and Lemma 2.5 in [9] with respectively Lemma 4, Lemma 5 and Lemma 2 above. ■

Remark 7. For simplicity, we restricted ourselves in this section to nested subspaces of $\mathring{\mathcal{S}}$ of the form $\mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$, with $|p(n) - q(n)| \leqslant 1$. However, the results clearly remain valid for more arbitrary nondecreasing sequences of nonnegative integers $p(n)$ and $q(n)$, satisfying (3.1), when redefining $\hat{\mathring{\mathcal{L}}}_n$ in (2.5) as $\hat{\mathring{\mathcal{L}}}_n = \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)}$ (as has been done in [9]).

4. Rational interpolation and $L_2^{\mu}(I)$ convergence. In this section we are mainly concerned with the $L_2^{\mu}(I)$ convergence when considering sequences of interpolating rational functions of increasing degree to a function $f(x)$ defined on I. This will be done by passing from the interval to the unit circle by means of the Joukowski Transformation $x = J(z)$, and using the result for the $L_2^{\mathring{\mu}}(\mathbb{T})$ convergence in Theorem 6 when considering sequences of interpolating rational functions of increasing degree to a function $\mathring{f}(z)$ defined on $\mathbb{T}$. We will start with the case in which $s = 1$, since this is the most easy case. We then have the following theorem.

THEOREM 8. *Consider the set of $n(1)$ distinct nodes $\mathbf{x}^{[\mathbf{1}]}_{\mathbf{n}(\mathbf{1})} \subset I$. Let μ be a positive measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by the relation (2.6). Suppose R^f_{n-1} is the unique rational function in $\mathcal{L}_{n-1}$ that interpolates f at the nodes $\mathbf{x}^{[\mathbf{1}]}_{\mathbf{n}(\mathbf{1})}$. Next, let the symbol '$\iota$' refer to either '$c$' or to '$*$', and assume $S^{\mathring{f}\,[\iota]}_{2n-2}$ is the unique rational function in $\mathring{\mathcal{S}}^{[\iota]}_{2n-2}$ that interpolates $\mathring{f}(z) = (f \circ J)(z)$ at the corresponding $N(1)$ distinct nodes $\mathbf{z}^{[\mathbf{1}]}_{\mathbf{N}(\mathbf{1})} \subset \mathbb{T}$. For any function f bounded on I it then holds that*

$$\left\| f - R^f_{n-1} \right\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - S^{\mathring{f}\,[\iota]}_{2n-2} \right\|_{\mathring{\mu},2}.$$

Proof. First, note that for f bounded on I, the interpolating rational function R^f_{n-1} always exists and is unique in $\mathcal{L}_{n-1}$. Clearly, the corresponding function $\mathring{f}$ is bounded too on $\mathbb{T}$, so that the interpolating rational function $S^{\mathring{f}\,[\iota]}_{2n-2}$ again always exists and is unique in $\mathring{\mathcal{S}}^{[\iota]}_{2n-2}$. Thus, let $\mathring{R}^f_{n-1}(z) = \left(R^f_{n-1} \circ J\right)(z)$. Then it follows from (2.7) that

$$\left\| f - R^f_{n-1} \right\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - \mathring{R}^f_{n-1} \right\|_{\mathring{\mu},2}.$$

Moreover, $\mathring{R}^f_{n-1}(z)$ is a rational function in $\mathring{\mathcal{S}}^{[\iota]}_{2n-2}$ that interpolates $\mathring{f}$ too at the nodes $\mathbf{z}^{[\mathbf{1}]}_{\mathbf{N}(\mathbf{1})} \subset \mathbb{T}$. Due to the uniqueness, it follows that $\mathring{R}^f_{n-1}(z) = S^{\mathring{f}\,[\iota]}_{2n-2}(z)$, which ends the proof. ■

Next, consider the case in which the interpolation nodes $\mathbf{x}^{[\mathbf{1}]}_{\mathbf{n}(\mathbf{1})}$ are $(\mu, 1)$-nodes. The following theorem then gives a convergence result in the L^μ_2-norm $\left\| f - R^f_{n-1} \right\|_{\mu,2}$ for n tending to infinity.

THEOREM 9. *Let μ be a positive measure on I, and assume $\sum_{j=1}^{\infty}(1-\left|J^{inv}(\alpha_j)\right|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists, the sequence of interpolating rational functions $\left\{R^f_{n-1}\right\}_{n=1}^{\infty}$, with $R^f_{n-1} \in \mathcal{L}_{n-1}$, at the $(\mu, 1)$-nodes $\mathbf{x}^{[\mathbf{1}]}_{\mathbf{n}(\mathbf{1})}$, $n = 1, 2, \ldots$, converge to f in L^μ_2-norm; i.e;*

$$\lim_{n\to\infty} \left\| f - R^f_{n-1} \right\|_{\mu,2} = 0.$$

Proof. From Theorem 8 it follows that

$$\lim_{n\to\infty} \left\| f - R^f_{n-1} \right\|_{\mu,2} = \frac{1}{\sqrt{2}} \lim_{n\to\infty} \left\| \mathring{f} - S^{\mathring{f}\,[\iota]}_{2n-2} \right\|_{\mathring{\mu},2},$$

where the symbol 'ι' refers to either 'c' or to '$*$'. Further, since the corresponding sets of $N(1)$ distinct nodes $\mathbf{z}^{[\mathbf{1}]}_{\mathbf{N}(\mathbf{1})} \subset \mathbb{T}$ are $(\mathring{\mu}, 1)$-nodes (see Theorem 1), it follows from Theorem 6 that

$$\lim_{n\to\infty} \left\| \mathring{f} - S^{\mathring{f}\,[\iota]}_{2n-2} \right\|_{\mathring{\mu},2} = 0.$$

This concludes the proof. ■

Next, we consider the case in which $s = 0$. For this, let us define the auxiliary subspaces

$$\mathring{\mathcal{S}}_{2k-1}^{[c,\beta]} = \mathring{\mathcal{L}}_k^{[\beta]c} \cdot \mathring{\mathcal{L}}_{(k-1)*} \quad \text{and} \quad \mathring{\mathcal{S}}_{2k}^{[c,\beta]} = \mathring{\mathcal{L}}_k^{[\beta]c} \cdot \mathring{\mathcal{L}}_{k*},$$

and

$$\mathring{\mathcal{S}}_{2k-1}^{[*,\beta]} = \mathring{\mathcal{L}}_{k-1}^{c} \cdot \mathring{\mathcal{L}}_{k*}^{[\beta]} \quad \text{and} \quad \mathring{\mathcal{S}}_{2k}^{[*,\beta]} = \mathring{\mathcal{L}}_k^{c} \cdot \mathring{\mathcal{L}}_{k*}^{[\beta]},$$

where $\beta \in \mathbb{D}$. Note that it holds for every $n > 0$ that $\mathring{\mathcal{S}}_{n-1}^{[c,\beta]} \subset \mathring{\mathcal{S}}_n^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{n-1}^{[*,\beta]} \subset \mathring{\mathcal{S}}_n^{[*,\beta]}$; thus, the results in the previous section remain valid for these auxiliary subspaces. We then have the following theorem.

THEOREM 10. *Consider the set of $n(0)$ distinct nodes $\mathbf{x}_{\mathbf{n(0)}}^{[\mathbf{0}]} \subset (-1,1)$. Let μ be a positive measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by the relation (2.6). Suppose R_{n-1}^f is the unique rational function in $\mathcal{L}_{n-1}$ that interpolates f at the nodes $\mathbf{x}_{\mathbf{n(0)}}^{[\mathbf{0}]}$. Next, suppose $\beta \in \mathbb{D}$ and assume $S_{2n-1}^{\mathring{f}\,[c,\beta]}$ and $S_{2n-1}^{\mathring{f}\,[*,\beta]}$ are the unique rational functions in respectively $\mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}$ that interpolate $\mathring{f}(z) = (f \circ J)(z)$ at the corresponding $N(0)$ distinct nodes $\mathbf{z}_{\mathbf{N(0)}}^{[\mathbf{0}]} \subset \mathbb{T} \setminus \{-1,1\}$. For any function f bounded on I it then holds that*

$$\left\| f - R_{n-1}^f \right\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[c,\beta]} \right\|_{\mathring{\mu},2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[*,\beta]} \right\|_{\mathring{\mu},2}.$$

Proof. First, note that for f bounded on I, the interpolating rational function R_{n-1}^f always exists and is unique in $\mathcal{L}_{n-1}$. Clearly, the corresponding function $\mathring{f}$ is bounded too on $\mathbb{T}$, so that the interpolating rational functions $S_{2n-1}^{\mathring{f}\,[c,\beta]}$ and $S_{2n-1}^{\mathring{f}\,[*,\beta]}$ again always exist and are unique in $\mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}$ respectively. Thus, let $\mathring{R}_{n-1}^f(z) = \left(R_{n-1}^f \circ J\right)(z)$. Then it follows from (2.7) that

$$\left\| f - R_{n-1}^f \right\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - \mathring{R}_{n-1}^f \right\|_{\mathring{\mu},2}.$$

Moreover, $\mathring{R}_{n-1}^f(z)$ is a rational function in $\mathring{\mathcal{S}}_{2n-2}$ that interpolates $\mathring{f}$ too at the nodes $\mathbf{z}_{\mathbf{N(0)}}^{[\mathbf{0}]} \subset \mathbb{T} \setminus \{-1,1\}$. Since $\mathring{\mathcal{S}}_{2n-2} \subset \mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{2n-2} \subset \mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}$, and due to the uniqueness, it follows that $\mathring{R}_{n-1}^f(z) = S_{2n-1}^{\mathring{f}\,[c,\beta]}(z) = S_{2n-1}^{\mathring{f}\,[*,\beta]}(z)$, which ends the proof. ■

Consider now the case in which the interpolation nodes $\mathbf{x}_{\mathbf{n(0)}}^{[\mathbf{0}]}$ are $(\mu,0,\alpha)$-nodes. The following theorem then gives a convergence result in the L_2^μ-norm $\left\| f - R_{n-1}^f \right\|_{\mu,2}$ for n tending to infinity.

THEOREM 11. *Let μ be a positive measure on I. Suppose $\alpha \in \overline{\mathbb{R}}_I$ and assume $\sum_{j=1}^{\infty}(1 - \left|J^{inv}(\alpha_j)\right|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists, the sequence of interpolating rational functions $\left\{R_{n-1}^f\right\}_{n=1}^{\infty}$, with $R_{n-1}^f \in \mathcal{L}_{n-1}$, at the $(\mu,0,\alpha)$-nodes $\mathbf{x}_{\mathbf{n(0)}}^{[\mathbf{0}]}$, $n = 1,2,\ldots$, converge to f in L_2^μ-norm; i.e;*

$$\lim_{n\to\infty} \left\| f - R_{n-1}^f \right\|_{\mu,2} = 0.$$

Proof. From Theorem 10 it follows that

$$\lim_{n\to\infty}\left\|f-R_{n-1}^{f}\right\|_{\mu,2}=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\left\|\mathring{f}-S_{2n-1}^{\mathring{f}\,[\iota,\beta]}\right\|_{\mathring{\mu},2},\quad \beta=J^{inv}(\alpha)\in(-1,1),$$

where the symbol 'ι' refers to either 'c' or to '$*$'. Further, since the corresponding sets of $N(0)$ distinct nodes $\mathbf{z}_{\mathbf{N(0)}}^{[\mathbf{0}]}\subset\mathbb{T}$ are $(\mathring{\mu},0,\beta)$-nodes (see Theorem 1), it follows from Theorem 6 that

$$\lim_{n\to\infty}\left\|\mathring{f}-S_{2n-1}^{\mathring{f}\,[\iota,\beta]}\right\|_{\mathring{\mu},2}=0.$$

This concludes the proof. ∎

Finally, let us consider the case in which $s=2$. As already pointed out in Section 2, an $(n+1)$-point rational Gauss-Lobatto quadrature surely does not exist whenever the last pole $\alpha_n\notin\overline{\mathbb{R}}$. To overcome this problem, we will work with the sequence of poles $\tilde{\mathcal{A}}=\{\alpha,\alpha_1,\alpha_2,\ldots\}\subset\overline{\mathbb{C}}_I$ instead, where $\alpha\in\overline{\mathbb{C}}_I$. Note that the interpolating rational functions do not depend on the order of the poles. So, we may as well work with nested spaces of rational functions $\mathcal{L}_n^{[\alpha]}$. We then have the following theorem.

THEOREM 12. *Consider the set of $n(2)$ distinct nodes $\mathbf{x}_{\mathbf{n(2)}}^{[\mathbf{2}]}\subset I$. Let μ be a positive measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by the relation (2.6). Suppose R_n^f is the unique rational function in $\mathcal{L}_n^{[\alpha]}$, with $\alpha\in\overline{\mathbb{C}}_I$, that interpolates f at the nodes $\mathbf{x}_{\mathbf{n(2)}}^{[\mathbf{2}]}$. Next, assume $S_{2n-1}^{\mathring{f}\,[c,\beta]}$ and $S_{2n-1}^{\mathring{f}\,[*,\beta]}$, with $\beta=J^{inv}(\alpha)\in\mathbb{D}$, are the unique rational functions in $\mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}$ respectively that interpolate $\mathring{f}(z)=(f\circ J)(z)$ at the corresponding $N(2)$ distinct nodes $\mathbf{z}_{\mathbf{N(2)}}^{[\mathbf{2}]}\subset\mathbb{T}$. For any function f bounded on I it then holds that*

$$\left\|f-R_n^f\right\|_{\mu,2}\leqslant\frac{1}{\sqrt{2}}\left\|\mathring{f}-S_{2n-1}^{\mathring{f}\,[c,\beta]}\right\|_{\mathring{\mu},2}\quad\text{and}\quad\left\|f-R_n^f\right\|_{\mu,2}\leqslant\frac{1}{\sqrt{2}}\left\|\mathring{f}-S_{2n-1}^{\mathring{f}\,[*,\beta]}\right\|_{\mathring{\mu},2}.$$

Proof. First, note that for f bounded on I, the interpolating rational function R_n^f always exists and is unique in $\mathcal{L}_n^{[\alpha]}$. Clearly, the corresponding function $\mathring{f}$ is bounded too on $\mathbb{T}$, so that the interpolating rational functions $S_{2n-1}^{\mathring{f}\,[c,\beta]}$ and $S_{2n-1}^{\mathring{f}\,[*,\beta]}$ again always exist and are unique in $\mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$ and $\mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}$ respectively. Thus, let $\mathring{R}_n^f(z)=\left(R_n^f\circ J\right)(z)$. Then it follows from (2.7) that

$$\left\|f-R_n^f\right\|_{\mu,2}=\frac{1}{\sqrt{2}}\left\|\mathring{f}-\mathring{R}_n^f\right\|_{\mathring{\mu},2}.$$

Moreover, $\mathring{R}_n^f(z)$ is a rational function in $\mathring{\mathcal{S}}_{2n}^{[\beta,\beta]}:=\mathring{\mathcal{L}}_n^{[\beta]c}\cdot\mathring{\mathcal{L}}_{n*}^{[\beta]}$ that interpolates $\mathring{f}$ too at the nodes $\mathbf{z}_{\mathbf{N(2)}}^{[\mathbf{2}]}\subset\mathbb{T}$. Note that $\mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}\subset\mathring{\mathcal{S}}_{2n}^{[\beta,\beta]}$ and $\mathring{\mathcal{S}}_{2n-1}^{[*,\beta]}\subset\mathring{\mathcal{S}}_{2n}^{[\beta,\beta]}$. So, let us define now

$$S_{2n}^{\mathring{f}}(z)=\frac{1}{2}\left\{S_{2n-1}^{\mathring{f}\,[c,\beta]}(z)+S_{2n-1}^{\mathring{f}\,[*,\beta]}(z)\right\}\in\mathring{\mathcal{S}}_{2n}^{[\beta,\beta]}.$$

Clearly, we then have that $S_{2n}^{\mathring{f}}(z_j)=\mathring{f}(z_j)$ for every $z_j\in\mathbf{z}_{\mathbf{N(2)}}^{[\mathbf{2}]}$ and $S_{2n}^{\mathring{f}}(z^{-1})=S_{2n}^{\mathring{f}}(z)$ for every $z\in\overline{\mathbb{C}}$. Consequently, there exists a function $F_n\in\mathcal{L}_n^{[\alpha]}$ so that

$S_{2n}^{\mathring{f}}(z) = (F_n \circ J)(z)$ and $F_n(x_j) = f(x_j)$ for every $x_j \in \mathbf{x}_{\mathbf{n(2)}}^{[\mathbf{2}]}$. Since the interpolating rational function $R_n^f \in \mathcal{L}_n^{[\alpha]}$ is unique, it follows that $F_n(x) = R_n^f(x)$, and hence, $\mathring{R}_n^f(z) = S_{2n}^{\mathring{f}}(z)$. Consequently,

$$\left\| \mathring{f} - \mathring{R}_n^f \right\|_{\mathring{\mu},2} = \left\| \mathring{f} - S_{2n}^{\mathring{f}} \right\|_{\mathring{\mu},2} = \left\| \frac{1}{2}(\mathring{f} + \mathring{f}) - \frac{1}{2}\{S_{2n-1}^{\mathring{f}\,[c,\beta]} + S_{2n-1}^{\mathring{f}\,[*,\beta]}\} \right\|_{\mathring{\mu},2}$$
$$\leqslant \frac{1}{2}\left\{ \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[c,\beta]} \right\|_{\mathring{\mu},2} + \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[*,\beta]} \right\|_{\mathring{\mu},2} \right\}.$$

Finally, note that $S_{(2n-1)*}^{\mathring{f}\,[*,\beta]\,c}(z) = S_{2n-1}^{\mathring{f}\,[c,\beta]}(z)$ due to the fact that $\mathring{\mathcal{S}}_{(2n-1)*}^{[*,\beta]\,c} = \mathring{\mathcal{S}}_{2n-1}^{[c,\beta]}$. Consequently, since the measure $\mathring{\mu}$ is symmetric, it holds that $\left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[c,\beta]} \right\|_{\mathring{\mu},2} = \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[*,\beta]} \right\|_{\mathring{\mu},2}$. This ends the proof. ■

Consider now the case in which the interpolation nodes $\mathbf{x}_{\mathbf{n(2)}}^{[\mathbf{2}]}$ are $(\mu, 2, \alpha)$-nodes. The following theorem then gives a convergence result in the L_2^μ-norm $\left\| f - R_n^f \right\|_{\mu,2}$ for n tending to infinity.

Theorem 13. *Let μ be a positive measure on I. Suppose $\alpha \in \overline{\mathbb{R}}_I$ and assume $\sum_{j=1}^\infty (1 - \left|J^{inv}(\alpha_j)\right|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists, the sequence of interpolating rational functions $\left\{R_n^f\right\}_{n=1}^\infty$, with $R_n^f \in \mathcal{L}_n^{[\alpha]}$, at the $(\mu, 2, \alpha)$-nodes $\mathbf{x}_{\mathbf{n(2)}}^{[\mathbf{2}]}$, $n = 1, 2, \ldots$, converge to f in L_2^μ-norm; i.e;*

$$\lim_{n\to\infty} \left\| f - R_n^f \right\|_{\mu,2} = 0.$$

Proof. From Theorem 12 it follows that

$$\lim_{n\to\infty} \left\| f - R_n^f \right\|_{\mu,2} \leqslant \frac{1}{\sqrt{2}} \lim_{n\to\infty} \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[\iota,\beta]} \right\|_{\mathring{\mu},2}, \quad \beta = J^{inv}(\alpha) \in (-1, 1),$$

where the symbol 'ι' refers to either 'c' or to '$*$'. Further, since the corresponding sets of $N(2)$ distinct nodes $\mathbf{z}_{\mathbf{N(2)}}^{[\mathbf{2}]} \subset \mathbb{T}$ are $(\mathring{\mu}, 2, \beta)$-nodes (see Theorem 1), it follows from Theorem 6 that

$$\lim_{n\to\infty} \left\| \mathring{f} - S_{2n-1}^{\mathring{f}\,[\iota,\beta]} \right\|_{\mathring{\mu},2} = 0.$$

This concludes the proof. ■

We conclude this section with the following density result.

Theorem 14. *Let μ be a positive measure on I. Then $\mathcal{L}$ is dense in $L_2^\mu(I)$ if $\sum_{j=1}^\infty (1 - \left|J^{inv}(\alpha_j)\right|) = \infty$.*

Proof. Note that for every $f \in L_2^\mu(I)$ it holds that the function $\mathring{f}(z) = (f \circ J)(z) = \mathring{f}_*^c(z) \in L_2^{\mathring{\mu}}(\mathbb{T})$. Further, with $\beta_j = J^{inv}(\alpha_j) \in \mathbb{D}$ for every $j > 0$ it follows from Lemma 3 that $\mathring{\mathcal{S}}$ is dense in the class $C(\mathbb{T})$ of 2π-periodic functions on $\mathbb{T}$ if $\sum_{j=1}^\infty (1 - |\beta_j|) = \infty$. On the other hand, the class $C(\mathbb{T})$ is also dense in $L_2^{\mathring{\mu}}(\mathbb{T})$ (see e.g. [13]). Thus for $\mathring{f} \in L_2^{\mathring{\mu}}(\mathbb{T})$ and any $\epsilon > 0$, there exists a function $\mathring{h} \in C(\mathbb{T})$ such that

$$\left\| \mathring{f} - \mathring{h} \right\|_{\mathring{\mu},2} < \epsilon/\sqrt{2}.$$

Consider now the function $h \in C(I)$ (i.e.; the class of continuous functions on I) such that $(h \circ J)(z) = [\mathring{h}(z) + \mathring{h}_*^c(z)]/2$. Then it holds that

$$\begin{aligned} \|f - h\|_{\mu,2} &= \frac{1}{\sqrt{2}} \left\| \frac{\mathring{f} + \mathring{f}_*^c}{2} - \frac{\mathring{h} + \mathring{h}_*^c}{2} \right\|_{\mathring{\mu},2} \\ &\leqslant \frac{1}{2\sqrt{2}} \left\{ \left\| \mathring{f} - \mathring{h} \right\|_{\mathring{\mu},2} + \left\| \mathring{f}_*^c - \mathring{h}_*^c \right\|_{\mathring{\mu},2} \right\} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - \mathring{h} \right\|_{\mathring{\mu},2} < \epsilon/2. \end{aligned}$$

Furthermore, there exists a function $\mathring{R} \in \mathring{\mathcal{S}}$ so that

$$\left\| \mathring{h} - \mathring{R} \right\|_{\mathring{\mu},2} < \epsilon/\sqrt{2}.$$

Thus, let $R \in \mathcal{L}$ be such that $(R \circ J)(z) = [\mathring{R}(z) + \mathring{R}_*^c(z)]/2$, then

$$\begin{aligned} \|h - R\|_{\mu,2} &= \frac{1}{\sqrt{2}} \left\| \frac{\mathring{h} + \mathring{h}_*^c}{2} - \frac{\mathring{R} + \mathring{R}_*^c}{2} \right\|_{\mathring{\mu},2} \\ &\leqslant \frac{1}{2\sqrt{2}} \left\{ \left\| \mathring{h} - \mathring{R} \right\|_{\mathring{\mu},2} + \left\| \mathring{h}_*^c - \mathring{R}_*^c \right\|_{\mathring{\mu},2} \right\} = \frac{1}{\sqrt{2}} \left\| \mathring{h} - \mathring{R} \right\|_{\mathring{\mu},2} < \epsilon/2. \end{aligned}$$

As a result,

$$\|f - R\|_{\mu,2} \leqslant \|f - h\|_{\mu,2} + \|h - R\|_{\mu,2} < \epsilon.$$

This proves the theorem. ■

As a consequence of the previous theorem, we have the following. (The proof is exactly the same as the one of [8, Cor. 4.2], and hence, we omit it.)

COROLLARY 15. *Let P_n^f denote the nth partial sum of the expansion of a function $f \in L_2^\mu(I)$ with respect to any orthonormal basis of $\mathcal{L}$ and let $\sum_{j=1}^\infty (1 - |J^{inv}(\alpha_j)|) = \infty$. Then $\lim_{n\to\infty} \left\| f - P_n^f \right\|_{\mu,2} = 0$.*

5. An application to rational interpolatory quadrature rules. In this section we will be concerned with the approximation of $J_\sigma(f)$ by means of an $n(s)$-point rational interpolatory quadrature $J_{n(s)}^\sigma(f)$ based on a preassigned set of $n(s)$ distinct nodes $\mathbf{x}_{n(s)}^{[s]}$ on I, where $s \in \{0, 1, 2\}$ and f is bounded on I. Note that it is always possible to determine the weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ in such a way that the approximation is exact for every function $f \in \mathcal{R}_{p,q} = \mathcal{L}_p \cdot \mathcal{L}_q^c$, where $p + q = n(s) - 1$. Clearly, the most general situation is obtained by setting $p = n(s) - 1$ and $q = 0$, so that

$$\mathcal{R}_{n(s)-1,0} = \breve{\mathcal{L}}_{n(s)-1} = \begin{cases} \breve{\mathcal{L}}_{n-1} := \mathcal{L}_{n-1}, & s < 2 \\ \breve{\mathcal{L}}_n := \mathcal{L}_n^{[\alpha]}, & s = 2, \end{cases} \tag{5.1}$$

where $\alpha \in \overline{\mathbb{C}}_I$; hence, possibly equal to α_n. In this case, the weights are determined by requiring that

$$J^{\sigma}_{n(s)}(f) = J_{\sigma}(\breve{R}^{f}_{n(s)-1}),$$

where $\breve{R}^{f}_{n(s)-1}$ denotes the unique rational function in $\breve{\mathcal{L}}_{n(s)-1}$ that interpolates the function f at the preassigned set of nodes. The success of such quadrature rules not only depends on the choice of nodes (see e.g. [22]), but also on the choice of poles. In this paper, however, we will assume that the poles are fixed in advance, and hence, we will only be concerned with the choice of nodes.

Let $\mathbf{z}^{[s]}_{N(s)}$ be the corresponding set of $N(s)$ distinct nodes on $\mathbb{T}$, with

$$\begin{array}{l} z_k = \overline{z}_{n-\lceil s/2\rceil+k} = J^{inv}(x_k) \in \mathbb{T}\setminus\{-1,1\}, \quad k=1,\ldots,(n-\lceil s/2\rceil) \\ z_{2n-1} = \pm 1 = x_n \quad \text{if} \quad s\neq 0, \quad \text{and} \quad z_{2n} = -z_{2n-1} = x_{n+1} \quad \text{if} \quad s=2. \end{array} \tag{5.2}$$

Suppose the spaces of rational functions $\breve{\mathring{\mathcal{S}}}^{[\iota]}_k$ (where the symbol 'ι' refers to either 'c' or to '$*$') are defined by

$$\breve{\mathring{\mathcal{S}}}^{[\iota]}_{2n-2} = \mathring{\mathcal{S}}_{2n-2}, \quad \breve{\mathring{\mathcal{S}}}^{[\iota]}_{2n-1} = \mathring{\mathcal{S}}^{[\iota,\beta]}_{2n-1} \quad \text{and} \quad \breve{\mathring{\mathcal{S}}}^{[\iota]}_{2n} = \mathring{\mathcal{S}}^{[\beta,\beta]}_{2n},$$

where $\beta = J^{inv}(\alpha)$. Further, let the measure $\mathring{\sigma}$ on $\mathbb{T}$ be related to the measure σ on I by means of (2.6), and suppose $\mathring{f}$ is bounded on $\mathbb{T}$. We then can consider the approximation of $I_{\mathring{\sigma}}(\mathring{f})$ by means of the following $N(s)$-point quadrature rules:

$$I^{\mathring{\sigma}}_{N(s)}(\mathring{f}) = \sum_{k=1}^{N(s)} \mathring{\lambda}^{\mathring{\sigma}}_k \mathring{f}(z_k) = I_{\mathring{\sigma}}(\mathring{f}), \quad \forall \mathring{f} \in \breve{\mathring{\mathcal{S}}}^{[c]}_{N(s)-1} \tag{5.3}$$

and

$$\tilde{I}^{\mathring{\sigma}}_{N(s)}(\mathring{f}) = \sum_{k=1}^{N(s)} \tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k \mathring{f}(z_k) = I_{\mathring{\sigma}}(\mathring{f}), \quad \forall \mathring{f} \in \breve{\mathring{\mathcal{S}}}^{[*]}_{N(s)-1}. \tag{5.4}$$

Note that these quadrature rules are of interpolatory type too, and are obtained by determining the weights $\mathring{\lambda}^{\mathring{\sigma}}_k$ and $\tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k$ in such a way that

$$I^{\mathring{\sigma}}_{N(s)}(\mathring{f}) = I_{\mathring{\sigma}}(\breve{S}^{\mathring{f}\,[c]}_{N(s)-1}) \quad \text{and} \quad \tilde{I}^{\mathring{\sigma}}_{N(s)}(\mathring{f}) = I_{\mathring{\sigma}}(\breve{S}^{\mathring{f}\,[*]}_{N(s)-1}),$$

where $\breve{S}^{\mathring{f}\,[c]}_{N(s)-1}$ and $\breve{S}^{\mathring{f}\,[*]}_{N(s)-1}$ denote the unique rational functions in respectively $\breve{\mathring{\mathcal{S}}}^{[c]}_{N(s)-1}$ and $\breve{\mathring{\mathcal{S}}}^{[*]}_{N(s)-1}$ that interpolate the function $\mathring{f}$ at the preassigned set of nodes $\mathbf{z}^{[s]}_{N(s)}$. The following theorem now provides expressions for the weights $\{\lambda^{\sigma}_k\}_{k=1}^{n(s)}$ in terms of the weights $\{\mathring{\lambda}^{\mathring{\sigma}}_k\}_{k=1}^{N(s)}$ and $\{\tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k\}_{k=1}^{N(s)}$.

THEOREM 16. *Let $\mathbf{x}^{[s]}_{n(s)}$ be a preassigned set of $n(s)$ distinct nodes on I, and let $\mathbf{z}^{[s]}_{N(s)}$ be the corresponding set of $N(s)$ distinct nodes on $\mathbb{T}$, given by* (5.2). *Consider the rational interpolatory quadrature $J^{\sigma}_{n(s)}(f)$ for $J_{\sigma}(f)$, based on the set of nodes $\mathbf{x}^{[s]}_{n(s)}$, where the weights $\{\lambda^{\sigma}_k\}_{k=1}^{n(s)}$ are chosen in such a way that the approximation is exact for every $f \in \breve{\mathcal{L}}_{n(s)-1}$, given by* (5.1). *Then for $s=0$ it holds that*

$$\lambda^{\sigma}_k = \frac{\mathring{\lambda}^{\mathring{\sigma}}_k + \mathring{\lambda}^{\mathring{\sigma}}_{n+k}}{2} = \frac{\tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k + \tilde{\mathring{\lambda}}^{\mathring{\sigma}}_{n+k}}{2}, \quad k=1,\ldots,n, \tag{5.5}$$

while for $s \in \{1,2\}$ it holds that

$$\begin{array}{l} \lambda_k^\sigma = \mathring{\lambda}_k^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n-1+k}^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_{n-1+k}^{\mathring{\sigma}}, \quad k = 1, \ldots, n-1 \\ \lambda_n^\sigma = \frac{\mathring{\lambda}_{2n-1}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_{2n-1}^{\mathring{\sigma}}}{2}, \quad \text{and} \quad \lambda_{n+1}^\sigma = \frac{\mathring{\lambda}_{2n}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_{2n}^{\mathring{\sigma}}}{2} \quad \text{if} \quad s = 2, \end{array}$$

where $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$ are the set of weights in respectively the rational interpolatory quadratures $I_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ for $I_{\mathring{\sigma}}(\mathring{f})$, based on the set of nodes $\mathbf{z}_{N(s)}^{[s]}$.

Proof. Let L_k, $k = 1, \ldots, n(s)$, denote the fundamental rational interpolating functions in $\breve{\mathcal{L}}_{n(s)-1}$, so that $L_k(x_j) = \delta_{k,j}$ for $j = 1, \ldots, n(s)$, and define $g_k(z) := (L_k \circ J)(z) \in \breve{\mathring{\mathcal{S}}}_{2[n(s)-1]}^{[c]} = \breve{\mathring{\mathcal{S}}}_{2[n(s)-1]}^{[*]}$. Then it holds that

$$\begin{array}{l} g_k(z_j) = g_k(z_{n-\lceil s/2 \rceil + j}) = \delta_{k,j}, \quad j = 1, \ldots, (n - \lceil s/2 \rceil) \\ g_k(z_{2n-1}) = \delta_{k,n} \quad \text{if} \quad s \neq 0, \quad \text{and} \quad g_k(z_{2n}) = \delta_{k,n+1} \quad \text{if} \quad s = 2, \end{array}$$

so that $\lambda_k^\sigma = J_\sigma(L_k) = \frac{1}{2} I_{\mathring{\sigma}}(g_k)$. Since for $s = 0$ it holds that g_k is in $\breve{\mathring{\mathcal{S}}}_{N(0)-1}^{[c]}$ as well as in $\breve{\mathring{\mathcal{S}}}_{N(0)-1}^{[*]}$, the equalities in (5.5) follow by applying the rational interpolatory quadratures (5.3) and (5.4) respectively.

Next, for $s \in \{1,2\}$ let $\mathring{l}_k$, $k = 1, \ldots, N(s)$, denote the fundamental rational interpolating functions in $\breve{\mathring{\mathcal{S}}}_{N(s)-1}^{[c]}$, so that $\mathring{l}_k(z_j) = \delta_{k,j}$ for $j = 1, \ldots, N(s)$, and define the rational functions $h_k \in \breve{\mathring{\mathcal{S}}}_{2[n(s)-1]}^{[c]} = \breve{\mathring{\mathcal{S}}}_{2[n(s)-1]}^{[*]}$ by

$$\begin{array}{l} h_k(z) = \mathring{l}_k(z) + \mathring{l}_{k*}^c(z) = \mathring{l}_{n-1+k}(z) + \mathring{l}_{(n-1+k)*}^c(z) = h_{n-1+k}(z), \quad k = 1, \ldots, n-1, \\ h_n(z) = \dfrac{\mathring{l}_{2n-1}(z) + \mathring{l}_{(2n-1)*}^c(z)}{2}, \quad \text{and} \quad h_{n+1}(z) = \dfrac{\mathring{l}_{2n}(z) + \mathring{l}_{(2n)*}^c(z)}{2} \quad \text{if} \quad s = 2. \end{array}$$

Clearly, we then have that $h_k(z) = g_k(z)$ for $k = 1, \ldots, n(s)$, so that

$$\lambda_k^\sigma = \left\{ \begin{array}{ll} \frac{1}{2}\left\{ I_{\mathring{\sigma}}(\mathring{l}_k) + I_{\mathring{\sigma}}(\mathring{l}_{k*}^c) \right\} = \frac{1}{2}\left\{ I_{\mathring{\sigma}}(\mathring{l}_{n-1+k}) + I_{\mathring{\sigma}}(\mathring{l}_{(n-1+k)*}^c) \right\}, & k < n \\ \frac{1}{4}\left\{ I_{\mathring{\sigma}}(\mathring{l}_{n-1+k}) + I_{\mathring{\sigma}}(\mathring{l}_{(n-1+k)*}^c) \right\}, & k \geqslant n. \end{array} \right.$$

Finally, since the measure $\mathring{\sigma}$ is symmetric, we have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} = I_{\mathring{\sigma}}(\mathring{l}_k) = I_{\mathring{\sigma}}(\mathring{l}_{k*}^c) = \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}, \quad k = 1, \ldots, N(s),$$

where the first and last equality follow by applying the rational interpolatory quadratures (5.3) and (5.4) respectively. This concludes the proof. ∎

From the previous theorem it follows that, for $s = 0$, we need to compute $2n$ weights in the rational interpolatory quadratures on $\mathbb{T}$ in order to obtain the n weights in the rational interpolatory quadrature on I. Under certain conditions on the nodes or on the measure σ and the poles $\{\alpha_1, \ldots, \alpha_{n-1}, \alpha\}$, this number can be reduced to n, as shown in the following two theorems.

Theorem 17. *Let $\mathbf{x}_{n(0)}^{[0]}$ be a preassigned set of $n(0)$ distinct nodes on $(-1,1)$, and let $\mathbf{z}_{N(0)}^{[0]}$ be the corresponding set of $N(0)$ distinct nodes on $\mathbb{T} \setminus \{-1,1\}$, given* *by* (5.2). *Consider the rational interpolatory quadrature* $J_n^\sigma(f)$ *for* $J_\sigma(f)$, *based on the set of nodes* $\mathbf{x}_{n(0)}^{[0]}$, *where the weights* $\{\lambda_k^\sigma\}_{k=1}^n$ *are chosen in such a way that the approximation is exact for every* $f \in \breve{\mathcal{L}}_{n-1}$, *given by* (5.1). *Define the rational function* $\psi_n \in \breve{\mathcal{L}}_n$ *in such a way that* $\psi_n(x_j) = 0$ *for every* $x_j \in \mathbf{x}_{n(0)}^{[0]}$, *and let* $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ *and* $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ *be the set of weights in respectively the rational interpolatory quadratures* $I_{2n}^{\mathring{\sigma}}(\mathring{f})$ *and* $\tilde{I}_{2n}^{\mathring{\sigma}}(\mathring{f})$ *for* $I_{\mathring{\sigma}}(\mathring{f})$, *based on the set of nodes* $\mathbf{z}_{N(0)}^{[0]}$. *Then it holds that*

$$\lambda_k^\sigma = \mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}} \quad \textit{and} \quad \lambda_k^\sigma = \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}, \quad k = 1, \dots, n \tag{5.6}$$

iff

$$J_\sigma(\psi_n) = 0. \tag{5.7}$$

Proof. (We will only prove the first equality in (5.6); the second equality can be proved in a similar way). Let $\mathring{l}_k$, $k = 1, \dots, 2n$, denote the fundamental rational interpolating functions in $\breve{\mathring{\mathcal{S}}}_{2n-1}^{[c]}$, so that $\mathring{l}_k(z_j) = \delta_{k,j}$ for $j = 1, \dots, 2n$. Then we have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \mathring{\lambda}_{n+k}^{\mathring{\sigma}} = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{n+k}) = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{(n+k)*}^c),$$

where the last equality follows from the fact that the measure $\mathring{\sigma}$ is symmetric. Note that, due to Theorem 16, it suffices to prove that $\mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}}$ for $k = 1, \dots, n$ iff $J_\sigma(\psi_n) = 0$. So, let us now define the rational function $\mathring{\psi}_{2n}(z) := (\psi_n \circ J)(z) = \mathring{\psi}_{(2n)*}^c(z)$. Then it is easily verified that

$$\mathring{l}_k(z) = \frac{(z-\beta)\mathring{\psi}_{2n}(z)}{(z_k - \beta)(z - z_k)\mathring{\psi}_{2n}'(z_k)},$$

where the prime denotes the derivative with respect to z. Further, note that from $\mathring{\psi}_{2n}(z) = \mathring{\psi}_{2n}(1/z)$ it follows that $\mathring{\psi}_{2n}'(z) = -\frac{\mathring{\psi}_{2n}'(1/z)}{z^2}$, so that

$$\begin{aligned} I_{\mathring{\sigma}}(\mathring{l}_{(n+k)*}^c) &= \int_{-\pi}^{\pi} \frac{(1-\beta z)\mathring{\psi}_{2n}(z)}{(z_{n+k} - \beta)(1 - z_{n+k}z)\mathring{\psi}_{2n}'(z_{n+k})} d\mathring{\sigma}(\theta) \\ &= \int_{-\pi}^{\pi} \frac{(1-\beta z)\mathring{\psi}_{2n}(z)}{(1 - \beta z_k)(z - z_k)\left[-\frac{\mathring{\psi}_{2n}'(1/z_k)}{z_k^2}\right]} d\mathring{\sigma}(\theta) \\ &= \int_{-\pi}^{\pi} \frac{(1-\beta z)\mathring{\psi}_{2n}(z)}{(1 - \beta z_k)(z - z_k)\mathring{\psi}_{2n}'(z_k)} d\mathring{\sigma}(\theta). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathring{\lambda}_k^{\mathring{\sigma}} - \mathring{\lambda}_{n+k}^{\mathring{\sigma}} &= \int_{-\pi}^{\pi} \frac{\mathring{\psi}_{2n}(z)}{(z - z_k)\mathring{\psi}_{2n}'(z_k)} \left[\frac{z-\beta}{z_k - \beta} - \frac{1-\beta z}{1 - \beta z_k}\right] d\mathring{\sigma}(\theta) \\ &= \frac{(1-\beta^2) I_{\mathring{\sigma}}(\mathring{\psi}_{2n})}{(z_k - \beta)(1 - \beta z_k)\mathring{\psi}_{2n}'(z_k)}. \end{aligned}$$

As a result, $\mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}}$ iff $I_{\mathring{\sigma}}(\mathring{\psi}_{2n}) = 0$; i.e.; iff $J_\sigma(\psi_n) = 0$. This ends the proof. ■

THEOREM 18. *Suppose σ is a real measure on I, and assume that $\alpha \in \overline{\mathbb{R}}_I$ and that the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$ are real and/or appear in complex conjugate pairs. Let $\mathbf{x}_{n(0)}^{[0]}$ be a preassigned set of $n(0)$ distinct nodes on I, and let $\mathbf{z}_{N(0)}^{[0]}$ be the corresponding set of $N(0)$ distinct nodes on $\mathbb{T}$, given by* (5.2)*. Consider the rational interpolatory quadrature $J_n^\sigma(f)$ for $J_\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(0)}^{[0]}$, where the weights $\{\lambda_k^\sigma\}_{k=1}^n$ are chosen in such a way that the approximation is exact for every $f \in \breve{\mathcal{L}}_{n-1}$, given by* (5.1)*. Then it holds that*

$$\lambda_k^\sigma = \Re\{\mathring{\lambda}_k^{\mathring{\sigma}}\} \quad \text{and} \quad \lambda_k^\sigma = \Re\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}, \quad k = 1, \ldots, n, \tag{5.8}$$

where $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ are the set of weights in respectively the rational interpolatory quadratures $I_{2n}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{2n}^{\mathring{\sigma}}(\mathring{f})$ for $I_{\mathring{\sigma}}(\mathring{f})$, based on the set of nodes $\mathbf{z}_{N(0)}^{[0]}$.

Proof. (We will only prove the first equality in (5.8); the second equality can be proved in a similar way). Let $\mathring{l}_k$, $k = 1, \ldots, 2n$, and $\mathring{\psi}_{2n}(z)$ be defined as before in the proof of the previous theorem. Note that, due to Theorem 16, it suffices to prove that $\mathring{\lambda}_k^{\mathring{\sigma}} = \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}}$ for $k = 1, \ldots, n$ under the conditions on the measure σ and the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$, given in the statement. We now have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} = I_{\mathring{\sigma}}(\mathring{l}_k) - \overline{I_{\mathring{\sigma}}(\mathring{l}_{n+k})} = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{n+k}^c),$$

where the last equality follows from the fact that the measure $\mathring{\sigma}$ is real symmetric. Further, we have that

$$\begin{aligned} I_{\mathring{\sigma}}(\mathring{l}_{n+k}^c) &= \int_{-\pi}^{\pi} \frac{(z - \overline{\beta})\mathring{\psi}_{2n}^c(z)}{(\overline{z}_{n+k} - \overline{\beta})(z - \overline{z}_{n+k})\overline{\mathring{\psi}'_{2n}(z_{n+k})}} d\mathring{\sigma}(\theta) \\ &= \int_{-\pi}^{\pi} \frac{(z - \beta)\overline{\mathring{\psi}_{2n}(z)}}{(z_k - \beta)(z - z_k)\overline{\left[-z_k^2\mathring{\psi}'_{2n}(z_k)\right]}} d\mathring{\sigma}(\theta). \end{aligned}$$

Consequently,

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} = \int_{-\pi}^{\pi} \frac{z_k(z - \beta)}{(z_k - \beta)(z - z_k)} 2\Re\left\{\overline{z}_k \frac{\mathring{\psi}_{2n}(z)}{\mathring{\psi}'_{2n}(z_k)}\right\} d\mathring{\sigma}(\theta).$$

Next, note that $\mathring{\psi}_{2n}(z)$ is of the form

$$\mathring{\psi}_{2n}(z) = \frac{c_n \prod_{j=1}^{2n}(z - z_j)}{(z - \beta)(1 - \beta z)\prod_{j=1}^{n-1}(1 - \beta_j z)(z - \beta_j)}, \quad c_n \in \mathbb{C}_0,$$

and that $\overline{\mathring{\psi}_{2n}(z)} = \frac{\overline{c}_n}{c_n}\mathring{\psi}_{2n}(z)$ due to the assumption on the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$. Without loss of generality, we may as well assume that $c_n = 1$, so that

$$\begin{aligned} \mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} &= 2\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}'_{2n}(z_k)}\right\} z_k \int_{-\pi}^{\pi} \frac{(z - \beta)\mathring{\psi}_{2n}(z)}{(z_k - \beta)(z - z_k)} d\mathring{\sigma}(\theta) \\ &= 2\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}'_{2n}(z_k)}\right\} z_k \mathring{\psi}'_{2n}(z_k) I_{\mathring{\sigma}}(\mathring{l}_k). \end{aligned}$$

Finally, it holds that

$$\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}_{2n}'(z_k)}\right\} = \frac{|z_k-\beta|^2\prod_{j=1}^{n-1}|z_k-\beta_j|^2}{\prod_{j=1,j\neq k}^{n}|z_k-z_j|^2}\Re\left\{\frac{1}{z_k-z_{n+k}}\right\} = 0.$$

■

Remark 19. Note that, if assumption (5.7) in Theorem 17 is satisfied, it holds that $J_\sigma(\varphi_n^{[\alpha]}) = 0 = J_n^\sigma(\varphi_n^{[\alpha]})$ so that $J_\sigma(f) = J_n^\sigma(f)$ for every $f \in \check{\mathcal{L}}_n = \mathcal{L}_n^{[\alpha]}$; see also [12, Rem. 4.7] for the polynomial case.

Remark 20. The condition on the poles $\{\alpha_1,\ldots,\alpha_{n-1}\}$ in Theorem 18 is sufficient but not necessary. A necessary but not sufficient condition is that the rational interpolatory quadrature $J_\sigma(f) \approx J_n^\sigma(f)$ is exact for every $f \in \tilde{\mathcal{L}}_m$ but not for $f \in \tilde{\mathcal{L}}_{m+1}\setminus\tilde{\mathcal{L}}_m$, where $\mathcal{L}_{n-1}\subseteq\tilde{\mathcal{L}}_m\subseteq\mathcal{L}_n\cdot\mathcal{L}_{n-1}^c$ and $\tilde{\mathcal{L}}_m^c = \tilde{\mathcal{L}}_m$.

So far, we have been mainly concerned with the algebraic aspects of the quadrature rules for $J_\sigma(f)$ and $I_{\mathring{\sigma}}(f)$, but nothing has been said yet about the "goodness" of these quadrature rules with respect to the numerical aspects. For this purpose, we will again consider de case in which the interpolation points $\mathbf{x}_{n(s)}^{[s]}$ on I are (μ,s,α)-nodes. In the remainder of this section, we will restrict ourselves to the case of the interval. Similar results are easily proved with the aid of [8, Sect. 3 and 4] and [17, Sect. 3] for the quadrature rules (5.3) and (5.4), based on an arbitrary sequence of $\mathring{\mu}$-nodes on $\mathbb{T}$ (and hence, for a complex measure $\mathring{\sigma}$ on $\mathbb{T}$ which is not necessarily symmetric). In fact, also here more general subspaces of $\mathring{\mathcal{S}}$, and of the form $\mathring{\mathcal{L}}_{p(n-1)}^c\cdot\mathring{\mathcal{L}}_{q(n-1)}$, with n the number of nodes in the rational interpolatory quadrature, can then be considered; see Remark 7.

In Theorem 23 we will prove – under certain conditions – the convergence of the rational interpolatory quadratures $J_{n(s)}^\sigma(f)$ for n tending to infinity. For this, we need the following two lemmas.

Lemma 21. *Suppose μ is a positive measure on I, and consider the sequence of orthonormal rational functions $\{\varphi_k\}_{k=0}^n$, with $\varphi_k\in\mathcal{L}_k\setminus\mathcal{L}_{k-1}$, so that $\varphi_k\perp_\mu\mathcal{L}_{k-1}$ and $\|\varphi_k\|_{\mu,2}=1$. Further, suppose $g(x)$ is a function on I such that $\|g\|_{\mu,2}<\infty$, and let P_n^g be its orthogonal projection in $L_2^\mu(I)$ onto $\mathcal{L}_n^c$; i.e.;*

$$P_n^g(x) = \sum_{k=0}^{n} c_k\varphi_k^c(x), \quad c_k = J_\mu(\varphi_k g).$$

Then it holds that $J_\mu(fP_n^g) = J_\mu(fg)$ for every $f\in\mathcal{L}_n$.

Proof. Since the sequence $\{\varphi_k\}_{k=0}^n$ forms an orthonormal basis for $\mathcal{L}_n$, it suffices to prove that

$$J_\mu(\varphi_j P_n^g) = J_\mu(\varphi_j g), \quad j=0,\ldots,n.$$

We now have for $j=0,\ldots,n$ that

$$J_\mu(\varphi_j P_n^g) = \sum_{k=0}^{n} c_k J_\mu(\varphi_j\varphi_k^c) = c_j = J_\mu(\varphi_j g).$$

This concludes the proof. ■

LEMMA 22. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that σ is absolutely continuous with respect to μ ($\sigma \ll \mu$) and*

$$\|g\|_{\mu,2} < \infty, \quad g(x) = \frac{d\sigma(x)}{d\mu(x)}. \tag{5.9}$$

Consider the set of (μ, s, α)-nodes $\mathbf{x}_{n(s)}^{[s]} \subset I$, where $s \in \{0,1,2\}$ and $\alpha \in \overline{\mathbb{R}}_I$, and let $\{\lambda_k^\mu\}_{k=1}^{n(s)}$ denote the corresponding weights in the rational Gauss-type quadrature. Let $a_{n,s}$ be given by

$$a_{n,s} = \begin{cases} 0, & s < 2 \\ \frac{J_\mu(\varphi_n^{[\alpha]} g)}{J_{n+1}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right)}, & s = 2, \end{cases}$$

where $\varphi_n^{[\alpha]} \in \mathcal{L}_n^{[\alpha]} \setminus \mathcal{L}_{n-1}$ is orthonormal to $\mathcal{L}_{n-1}$ with respect to the measure μ and inner product (2.2), and define the weights

$$\lambda_k^{\mathring{\sigma}} = \lambda_k^{\mathring{\mu}} \left[a_{n,s}\varphi_n^{[\alpha]c}(x_k) + P_{n-1}^g(x_k)\right], \quad k = 1, \ldots, n(s),$$

where P_{n-1}^g is the orthogonal projection in $L_2^\mu(I)$ of $g(x)$ onto $\mathcal{L}_{n-1}^c$. Then the rational interpolatory quadrature $J_\sigma(f) \approx J_{n(s)}^\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(s)}^{[s]}$ and weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ is exact for every $f \in \breve{\mathcal{L}}_{n(s)-1}$.

Proof. First, note that for $\sigma \ll \mu$, we have with $g(x) = \frac{d\sigma(x)}{d\mu(x)}$ that $J_\mu(fg) = J_\sigma(f)$ for every function f. So, consider now the case in which $f \in \mathcal{L}_{n-1}$. We then have that

$$\begin{aligned} J_{n(s)}^\sigma(f) &= a_{n,s} J_{n(s)}^\mu(f\varphi_n^{[\alpha]c}) + J_{n(s)}^\mu(fP_{n-1}^g) \\ &= a_{n,s} J_{n(s)}^\mu(f\varphi_n^{[\alpha]c}) + J_\mu\left(fP_{n-1}^g\right) = a_{n,s} J_{n(s)}^\mu(f\varphi_n^{[\alpha]c}) + J_\sigma(f), \end{aligned}$$

where the last equality follows from the previous lemma. Thus, we find that $J_{n(s)}^\sigma(f) = J_\sigma(f)$ iff $a_{n,s} J_{n(s)}^\mu(f\varphi_n^{[\alpha]c}) = 0$. Since $\alpha \in \overline{\mathbb{R}}_I$, we now have that $a_{n,s} J_{n(s)}^\mu(f\varphi_n^{[\alpha]c}) = 0$ iff $s \neq 1$ or $a_{n,1} = 0$. Hence, we have proved the statement for $s < 2$.

Consider now the case in which $s \in \{0, 2\}$ and $f = \varphi_n^{[\alpha]}$. It then holds that

$$\begin{aligned} J_{n(s)}^\sigma(\varphi_n^{[\alpha]}) &= a_{n,s} J_{n(s)}^\mu(\varphi_n^{[\alpha]}\varphi_n^{[\alpha]c}) + J_{n(s)}^\mu(\varphi_n^{[\alpha]}P_{n-1}^g) \\ &= a_{n,s} J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right) + J_\mu\left(\varphi_n^{[\alpha]}P_{n-1}^g\right) = a_{n,s} J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right). \end{aligned}$$

Note that $J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right) = 0$ for $s = 0$, due to the fact that $\varphi_n^{[\alpha]}(x_k) = 0$ for every $(\mu, 0, \alpha)$-node x_k. Consequently, the quadrature is exact for every $f \in \mathcal{L}_n^{[\alpha]}$ iff $J_\mu(\varphi_n^{[\alpha]} g) = J_\sigma(\varphi_n^{[\alpha]}) = 0$ (see also Remark 19). Thus, if $\breve{P}_n^g$ denotes the orthogonal projection in $L_2^\mu(I)$ of $g(x)$ onto $\breve{\mathcal{L}}_n^c$, then we deduce form the previous lemma that $\breve{P}_n^g(x) \equiv P_{n-1}^g(x)$ whenever $J_\mu(\varphi_n^{[\alpha]} g) = 0$. As a result, the statement remains valid for this special situation, and we may as well put $a_{n,0} = 0$ too.

Finally, for the case in which $s = 2$, it holds that $J_{n+1}^{\mu}\left(\left|\varphi_n^{[\alpha]}\right|^2\right) \in \mathbb{R}_0^+$. Thus, setting $a_{n,2} = J_\mu(\varphi_n^{[\alpha]} g)/J_{n+1}^{\mu}\left(\left|\varphi_n^{[\alpha]}\right|^2\right)$, we get that

$$J_{n+1}^{\sigma}(\varphi_n^{[\alpha]}) = \frac{J_\mu(\varphi_n^{[\alpha]} g)}{J_{n+1}^{\mu}\left(\left|\varphi_n^{[\alpha]}\right|^2\right)} J_{n+1}^{\mu}\left(\left|\varphi_n^{[\alpha]}\right|^2\right) = J_\mu(\varphi_n^{[\alpha]} g) = J_\sigma(\varphi_n^{[\alpha]}).$$

This concludes the proof. ∎

The following theorem now provides a convergence result for the rational interpolatory quadratures $J_{n(s)}^{\sigma}(f)$ for n tending to infinity.

THEOREM 23. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that $\sigma \ll \mu$ and $\|g\|_{\mu,2} < \infty$, where g is defined as before in (5.9). Assume $\sum_{j=1}^{\infty}(1 - \left|J^{inv}(\alpha_j)\right|) = \infty$, and consider the rational interpolatory quadratures $J_\sigma(f) \approx J_{n(s)}^{\sigma}(f)$, $n = 1, 2, \ldots$, based on the sets of (μ, s, α)-nodes $\mathbf{x}_{n(s)}^{[s]} \subset I$, where $s \in \{0, 1, 2\}$ and $\alpha \in \overline{\mathbb{R}}_I$. Further, let the weights in the quadratures be determined in such a way that the approximation is exact for every $f \in \check{\mathcal{L}}_{n(s)-1}$. Then it holds that $\lim_{n\to\infty} J_{n(s)}^{\sigma}(f) = J_\sigma(f)$ for all functions f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists.*

Proof. For every $n > 0$, let $\check{R}_{n(s)-1}^{f} \in \check{\mathcal{L}}_{n(s)-1}$ denote the interpolating rational function for f at the sets of nodes $\mathbf{x}_{n(s)}^{[s]}$. Since $J_{n(s)}^{\sigma}(f) = J_{n(s)}^{\sigma}(\check{R}_{n(s)-1}^{f}) = J_\sigma(\check{R}_{n(s)-1}^{f})$, we have that

$$\begin{aligned}\left|J_\sigma(f) - J_{n(s)}^{\sigma}(f)\right| &= \left|J_\sigma(f - \check{R}_{n(s)-1}^{f})\right| \\ &= \left|J_\mu\left([f - \check{R}_{n(s)-1}^{f}]g\right)\right| \leqslant J_\mu\left(\left|f - \check{R}_{n(s)-1}^{f}\right| |g|\right).\end{aligned}$$

Making use of the Cauchy–Schwarz inequality, and setting $\|g\|_{\mu,2} = M < \infty$, we obtain that

$$\left|J_\sigma(f) - J_{n(s)}^{\sigma}(f)\right| \leqslant M \cdot \left\|f - \check{R}_{n(s)-1}^{f}\right\|_{\mu,2},$$

where it follows from Theorems 9, 11 and 13 that

$$\lim_{n\to\infty}\left\|f - \check{R}_{n(s)-1}^{f}\right\|_{\mu,2} = 0.$$

∎

Under the same assumptions on the measure σ as in the previous theorem, it follows from the Banach-Steinhaus Theorem (see e.g. [21, Thm. 2.5]) that there exists a positive constant M_s so that for every $n > 0$, $S_n = \sum_{k=1}^{n(s)} |\lambda_k^{\sigma}| \leqslant M_s$. The next theorem implies as a special case (i.e., with $f(x) \equiv 1$) that the sequence S_n converges; namely that $\lim_{n\to\infty} S_n = \int_{-1}^{1} |d\sigma(x)|$. First, we need the following lemma.

LEMMA 24. *Suppose μ is a positive measure on I, and assume $\sum_{j=1}^{\infty}(1 - \left|J^{inv}(\alpha_j)\right|) = \infty$. For $s \in \{0, 1, 2\}$ and $n = 1, 2, \ldots$, let $J_{n(s)}^{\mu}(f)$ denote the rational Gauss-type quadrature based on the (μ, s, α)-nodes, where $\alpha \in \overline{\mathbb{R}}_I$. Then it holds that $\lim_{n\to\infty} J_{n(s)}^{\mu}(f) = J_\mu(f)$ for all μ-integrable functions f.*

Proof. The statement directly follows from Lemma 4 together with Theorem 1 and Lemma 2. ■

THEOREM 25. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that $\sigma \ll \mu$ and $\|g\|_{\mu,2} < \infty$, where g is defined as before in (5.9). Assume $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$, and consider the sets of (μ, s, α)-nodes $\mathbf{x}_{n(s)}^{[s]} \subset I$, $n = 1, 2, \ldots$, where $s \in \{0, 1, 2\}$ and $\alpha \in \overline{\mathbb{R}}_I$. Let the weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ be defined as before in Lemma 22, so that the corresponding rational interpolatory quadrature $J_\sigma(f) \approx J_{n(s)}^\sigma(f)$ is exact for every $f \in \breve{\mathcal{L}}_{n(s)-1}$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n(s)} |\lambda_k^\sigma| \, f(x_k) = J_\mu(f \cdot |g|), \tag{5.10}$$

for all functions f bounded on I, for which the function $f \cdot |g|$ is μ-integrable.

Proof. For ease of notation, we will denote the sum on the left hand side in (5.10) by $J_{n(s)}^\mu(f \cdot |g_n^\perp|)$, where $g_n^\perp(x) := a_{n,s}\varphi_n^{[\alpha]c}(x) + P_{n-1}^g(x) \in \breve{\mathcal{L}}_{n(s)-1}^c$; see Lemma 22.

Choose an arbitrary $\hat{g} \in \mathcal{L}_2^\mu(I)$ (i.e.; $\|\hat{g}\|_{\mu,2} < \infty$). By the triangle inequality and because the weights λ_k^σ depend linearly on σ, it follows that

$$\begin{aligned} T_n &:= \left| J_{n(s)}^\mu(f \cdot |g_n^\perp|) - J_\mu(f \cdot |g|) \right| \\ &\leqslant J_{n(s)}^\mu\left(|f| \cdot |g_n^\perp - \hat{g}_n^\perp|\right) + J_\mu\left(|f| \cdot |g - \hat{g}|\right) + \left| J_{n(s)}^\mu(f \cdot |\hat{g}_n^\perp|) - J_\mu(f \cdot |\hat{g}|) \right|, \end{aligned}$$

Since f is bounded on I, there exists a number M_f such that $|f(x)| \leqslant M_f < \infty$ for every $x \in I$. Hence,

$$\begin{aligned} T_n \leqslant M_f J_{n(s)}^\mu\left(|g_n^\perp - \hat{g}_n^\perp|\right) + M_f J_\mu\left(|g - \hat{g}|\right) + \left| J_{n(s)}^\mu(f \cdot |\hat{g}_n^\perp|) - J_\mu(f \cdot |\hat{g}|) \right| \\ = T_{n,1} + T_{n,2} + T_{n,3}, \end{aligned}$$

where $T_{n,1} = \left| J_{n(s)}^\mu(f \cdot |\hat{g}_n^\perp|) - J_\mu(f \cdot |\hat{g}|) \right|$, $T_{n,2} = M_f J_{n(s)}^\mu\left(|g_n^\perp - \hat{g}_n^\perp|\right)$, and $T_{n,3} = M_f J_\mu\left(|g - \hat{g}|\right)$. To estimate $T_{n,3}$, we use the Cauchy–Schwarz inequality to get

$$T_{n,3} \leqslant M_f \|g - \hat{g}\|_{\mu,2} \cdot K_\mu, \quad K_\mu := \|1\|_{\mu,2} < \infty.$$

For $T_{n,2}$ we note that for every function f_n of the form $f_n(x) = c_n\varphi_n^{[\alpha]c}(x) + f_{n-1}(x)$, with $f_{n-1} \in \mathcal{L}_{n-1}^c$, it holds that

$$\begin{aligned} J_{n(s)}^\mu\left(|f_n|\right) &\leqslant |c_n| \, J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]c}\right|\right) + J_{n(s)}^\mu\left(|f_{n-1}|\right) \\ &\leqslant \sqrt{\sum_{k=1}^{n(s)} \lambda_k^\mu} \cdot \left\{ |c_n| \sqrt{J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right)} + \sqrt{J_{n(s)}^\mu\left(|f_{n-1}|^2\right)} \right\} \\ &= K_\mu \cdot \left\{ |c_n| \sqrt{J_{n(s)}^\mu\left(\left|\varphi_n^{[\alpha]}\right|^2\right)} + \|f_{n-1}\|_{\mu,2} \right\}, \end{aligned}$$

where the last equality follows from the fact that the quadratures are all exact in $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$.

For our purpose, c_n equals $a_{n,s} - \hat{a}_{n,s}$. So, let us first consider the case in which $s < 2$. We then have that

$$T_{n,2} \leqslant M_f \left\| g_n^{\perp} - \hat{g}_n^{\perp} \right\|_{\mu,2} \cdot K_\mu \leqslant M_f \left\| g - \hat{g} \right\|_{\mu,2} \cdot K_\mu,$$

so that

$$T_n \leqslant T_{n,1} + 2M_f \left\| g - \hat{g} \right\|_{\mu,2} \cdot K_\mu.$$

Consider now the case in which $\hat{g} = P_N^g \in \mathcal{L}_N^c$. From Corollary 15 it then follows that for every $\epsilon' > 0$ there exists an integer l, so that for every $N \geqslant l$:

$$2M_f \left\| g - P_N^g \right\|_{\mu,2} \cdot K_\mu < \epsilon'/2,$$

Thus, for $n > N \geqslant l$ we get that

$$T_n < \epsilon'/2 + \left| J_{n(s)}^{\mu}(f \cdot |P_N^g|) - J_\mu(f \cdot |P_N^g|) \right|.$$

Further, since

$$\begin{aligned} |J_\mu(f \cdot |g|) - J_\mu(f \cdot |P_N^g|)| &\leqslant J_\mu(|f| \cdot |g - P_N^g|) \\ &\leqslant \|f\|_{\mu,2} \cdot \|g - P_N^g\|_{\mu,2} \leqslant M_f \|g - P_N^g\|_{\mu,2} \cdot K_\mu < \epsilon'/4, \end{aligned}$$

and $f \cdot |g|$ is assumed to be μ-integrable, it follows that $f \cdot |P_N^g|$ is μ-integrable too. Consequently, from Lemma 24 we deduce that for every $\epsilon' > 0$ there exists an integer k, so that for every $n \geqslant k$:

$$\left| J_{n(s)}^{\mu}(f \cdot |P_N^g|) - J_\mu(f \cdot |P_N^g|) \right| < \epsilon'/2.$$

Taking $n > \max\{N, k\}$ and setting $\epsilon = \epsilon'$, we obtain in this way that $T_n < \epsilon$. This proves the statement for $s < 2$.

Next, for $s = 2$ we have that

$$|c_n| \sqrt{J_{n+1}^{\mu}\left(\left| \varphi_n^{[\alpha]} \right|^2 \right)} = \frac{\left| J_\mu(\varphi_n^{[\alpha]}(g - \hat{g})) \right|}{\sqrt{J_{n+1}^{\mu}\left(\left| \varphi_n^{[\alpha]} \right|^2 \right)}} \leqslant \frac{\|g - \hat{g}\|_{\mu,2}}{\sqrt{J_{n+1}^{\mu}\left(\left| \varphi_n^{[\alpha]} \right|^2 \right)}},$$

so that

$$T_n \leqslant T_{n,1} + 2M_f \|g - \hat{g}\|_{\mu,2} \cdot K_\mu \left(1 + \frac{1}{2} \left[J_{n+1}^{\mu}\left(\left| \varphi_n^{[\alpha]} \right|^2 \right) \right]^{-1/2} \right).$$

Note that $J_\mu\left(\left| \varphi_n^{[\alpha]} \right|^2 \right) = 1$, and hence, from Lemma 24 we now deduce that that for every $\epsilon'' > 0$ there exists an integer m, so that for every $n \geqslant m$

$$1 - \epsilon'' < \left[J_{n+1}^{\mu}\left(\left| \varphi_n^{[\alpha]} \right|^2 \right) \right]^{-1/2} < 1 + \epsilon''.$$

Proceeding as before, with $\hat{g} = P_N^g$, and taking $n > \max\{N, k, m\}$, we obtain in this way that $T_n < \frac{\epsilon'}{4}(5 + \epsilon'') = \epsilon$. This concludes the proof. ■

6. Error bound and rate of convergence. In this section we will provide an error bound for the rational interpolatory quadrature formulas considered in the previous section. For this, we will again pass from the interval to the unit circle by means of the Joukowski Transformation $x = J(z) \in \mathbb{C}$. Setting $x = \Re\{x\} + \mathbf{i}\Im\{x\}$ and $z = \rho e^{\mathbf{i}\theta}$, we obtain that $\Re\{x\} + \mathbf{i}\Im\{x\} = J(\rho e^{\mathbf{i}\theta}) = \frac{1}{2}(\rho + \rho^{-1})\cos\theta + \mathbf{i}\frac{1}{2}(\rho - \rho^{-1})\sin\theta$. Thus, for $0 < \rho < 1$, the circles

$$\mathcal{C}_\rho := \{z \in \mathbb{C} : |z| = \rho\} \quad \text{and} \quad \mathcal{C}_{\frac{1}{\rho}} := \{z \in \mathbb{C} : |z| = \rho^{-1}\} \tag{6.1}$$

map onto the ellipse

$$\mathcal{E}_\rho := \left\{ x \in \mathbb{C} : \left(\frac{2\Re\{x\}}{\rho + \rho^{-1}} \right)^2 + \left(\frac{2\Im\{x\}}{\rho - \rho^{-1}} \right)^2 = 1 \right\}. \tag{6.2}$$

Let f be an analytic function on some open neighborhood of I, and suppose that μ is a positive measure on I. Consider an n-point quadrature rule $J_n^\mu(f) = \sum_{j=1}^n \lambda_k^\mu f(x_k)$ which is exact in the space $\tilde{\mathcal{L}}_m := \mathcal{L}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\} \subset \mathcal{L}_n^{[\alpha]} \cdot \mathcal{L}_{n-1}^c$, where $\alpha \in \overline{\mathbb{C}}_I$, and let $E_n^\mu(f) := J_\mu(f) - J_n^\mu(f)$ denote the error on the nth approximation. Further, suppose that U is the domain of analyticity of $\pi_m f$, where $\pi_m(x) = \prod_{j=1}^m (1 - x/\tilde{\alpha}_j)$, and let $0 < \rho < 1$ be such that $\mathcal{E}_\rho \subset U$. Then the following upper bound has been proved in [19, Eq. (5)]:

$$|E_n^\mu(f)| \leqslant \frac{\|\pi_m f\|_{\mathcal{E}_\rho}}{2\pi} \left(\int_{-1}^1 d\mu(x) + \sum_{k=1}^n |\lambda_k^\mu| \right) \int_{\mathcal{E}_\rho} e_{m+1}\left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) |dv|, \tag{6.3}$$

where

$$e_{m+1}\left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) = \min_{p \in \mathcal{P}_{m+1},\, p(v)=1} \left\| \frac{p(\cdot)}{\pi_m(\cdot)(v - \cdot)} \right\|_I$$

and $\|h\|_K$ stands for the maximum value of the continuous function $|h|$ on the compact set K. In [19, Sect. 3] the poles $\tilde{\alpha}_j$ where assumed to be real and/or appearing in complex conjugate pairs. However, following the steps in the proof of (6.3), it is easy to see that the expression for the upper bound remains valid without this assumption on the poles. Furthermore, the result is easily extended to the case of a complex measure σ just by replacing $d\mu(x)$ and λ_k^μ with $|d\sigma(x)|$ and λ_k^σ respectively; thus,

$$|E_n^\sigma(f)| \leqslant \frac{\|\pi_m f\|_{\mathcal{E}_\rho}}{2\pi} \left(\int_{-1}^1 |d\sigma(x)| + \sum_{k=1}^n |\lambda_k^\sigma| \right) \int_{\mathcal{E}_\rho} e_{m+1}\left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) |dv|.$$

We now have the following lemma, which is a generalization of [19, Lem. 1] to the case of arbitrary complex poles outside I.

LEMMA 26. *Suppose $x = J(z) \in I$, $v = J(w) \in \mathbb{C}$ and $\tilde{\alpha}_j = J(\tilde{\beta}_j) \in \overline{\mathbb{C}}_I$. Let us denote*

$$V_{m+1} = (1 + w^2) \prod_{j=1}^m (1 + \tilde{\beta}_j^2), \quad \mathring{\pi}_m(z) = \prod_{j=1}^m (z - \tilde{\beta}_j),$$

and set

$$\frac{\Phi_{m+1}(x)}{V_{m+1}} = \frac{\pi_m(x)(v - x)}{2^{m+1} v} \left\{ \frac{(z - \overline{w})\mathring{\pi}_m^c(z)}{(1 - zw) z^m \mathring{\pi}_{m*}^c(z)} + \frac{(1 - z\overline{w}) z^m \mathring{\pi}_{m*}(z)}{(z - w)\mathring{\pi}_m(z)} \right\}.$$

Then, $\Phi_{m+1}(x)$ is a polynomial in the variable x of degree $m + 1$.

Proof. First, note that

$$V_{m+1}\pi_m(x)(1-x/v) = \mathring{\pi}_m(z)\mathring{\pi}^c_{m*}(z)\frac{(z-w)(1-zw)}{z},$$

so that

$$2^{m+1}\Phi_{m+1}(x) = \frac{(z-w)(z-\overline{w})}{z^{m+1}}\mathring{\pi}_m(z)\mathring{\pi}^c_m(z) + z^{m-1}(1-zw)(1-z\overline{w})\mathring{\pi}^c_{m*}(z)\mathring{\pi}_{m*}(z).$$

Next, with $z = e^{\mathbf{i}\theta}$ we have

$$\frac{(z-w)(z-\overline{w})}{z^{m+1}}\mathring{\pi}_m(z)\mathring{\pi}^c_m(z) = \frac{(e^{\mathbf{i}\theta}-w)(e^{\mathbf{i}\theta}-\overline{w})}{e^{\mathbf{i}(m+1)\theta}}\prod_{j=1}^{m}(e^{\mathbf{i}\theta}-\tilde{\beta}_j)(e^{\mathbf{i}\theta}-\overline{\tilde{\beta}_j})$$

and

$$z^{m-1}(1-zw)(1-z\overline{w})\mathring{\pi}^c_{m*}(z)\mathring{\pi}_{m*}(z)$$
$$= e^{\mathbf{i}(m+1)\theta}\left(\frac{1}{e^{\mathbf{i}\theta}}-w\right)\left(\frac{1}{e^{\mathbf{i}\theta}}-\overline{w}\right)\prod_{j=1}^{m}\left(\frac{1}{e^{\mathbf{i}\theta}}-\tilde{\beta}_j\right)\left(\frac{1}{e^{\mathbf{i}\theta}}-\overline{\tilde{\beta}_j}\right).$$

Therefore, $\Phi_{m+1}(x)$ is a trigonometrical polynomial of degree $m+1$ at most, and hence, can be expressed as a linear combination of $\cos(k\theta)$, $k = 0,\dots,m+1$. An easy calculation shows that the coefficient of $\cos((m+1)\theta)$ is $(1+|w|^2\prod_{j=1}^{m}\left|\tilde{\beta}_j\right|^2)/2^m \neq 0$. This concludes the proof. ■

Note that for $|z| = 1$, it holds that $|(z-\overline{w})/(1-zw)| = |(1-z\overline{w})/(z-w)| = 1$, and that

$$\frac{\mathring{\pi}^c_m(z)}{z^m\mathring{\pi}^c_{m*}(z)} = \tilde{B}^c_m(z) \quad \text{and} \quad \frac{z^m\mathring{\pi}_{m*}(z)}{\mathring{\pi}_m(z)} = \tilde{B}_{m*}(z),$$

where $B_m(z)$ is the Blaschke product, given by (2.3). Hence, from the previous lemma it follows that

$$\left\|\frac{\Phi_{m+1}(\cdot)}{\pi_m(\cdot)(v-\cdot)}\right\|_I \leqslant \frac{|V_{m+1}|}{2^m\,|v|} = \frac{2\,|w|}{2^m}\prod_{j=1}^{m}\left|1+\tilde{\beta}_j^2\right|.$$

We are now in a position to prove the following generalization of [19, Thm. 1].

THEOREM 27. *Suppose σ is a complex measure on I. Let f be an analytic function on a neighborhood of the interval I, and suppose $J^\sigma_n(f) = \sum_{k=1}^{n}\lambda^\sigma_k f(x_k)$ is a quadrature rule for the integral $J_\sigma(f)$ that is exact in the space $\tilde{\mathcal{L}}_m := \mathcal{L}\{\tilde{\alpha}_1,\dots,\tilde{\alpha}_m\} \subseteq \mathcal{L}^{[\alpha]}_n \cdot \mathcal{L}^c_{n-1}$ (where $\alpha \in \overline{\mathbb{C}}_I$). Denote the error on the nth approximation by $E^\sigma_n(f) := J_\sigma(f) - J^\sigma_n(f)$. Define Υ_m and F_m by*

$$\Upsilon_m = \prod_{j=1}^{m}\left\{1+\left[J^{inv}(\tilde{\alpha}_j)\right]^2\right\} \quad \textit{and} \quad F_m(x) = f(x)\prod_{j=1}^{m}(1-x/\tilde{\alpha}_j),$$

and let U be the domain of analyticity of F_m. Further, let $\rho < 1$ be such that $\mathcal{E}_\rho \subset U$, where the ellipse $\mathcal{E}_\rho$ is defined as before in (6.2). *Then*

$$|E^\sigma_n(f)| \leqslant \left(\frac{2\rho^{m+1}}{1-\rho^2}\right)|\Upsilon_m|\,\|F_m\|_{\mathcal{E}_\rho}\left(\int_{-1}^{1}|d\sigma(x)| + \sum_{k=1}^{n}|\lambda^\sigma_k|\right)G_m(\rho), \tag{6.4}$$

where $\|F_m\|_{\mathcal{E}_\rho} := \max\{|F_m(x)| : x \in \mathcal{E}_\rho\}$, *and*

$$G_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{j=1}^{m}\frac{1}{|1-\rho J^{inv}(\tilde{\alpha}_j)e^{\mathbf{i}t}|^2}dt, \tag{6.5}$$

with the product in the integrand equal to 1 *when* $m = 0$.

Proof. Proceeding as in [19, Sect. 3], if we choose

$$p(x) = \frac{2^m\overline{w}^m}{1-|w|^2}\frac{1}{\sqrt{v^2-1}}\prod_{j=1}^{m}\left|1-w\tilde{\beta}_j\right|^{-2}\Phi_{m+1}(x),$$

then p is a polynomial of degree $m+1$ such that $p(v) = 1$ and

$$\left\|\frac{p(\cdot)}{\pi_m(\cdot)(v-\cdot)}\right\|_I \leqslant \frac{2|w|^{m+1}}{\left|\sqrt{v^2-1}\right|(1-|w|^2)}\prod_{j=1}^{m}\frac{\left|1+\tilde{\beta}_j^2\right|}{\left|1-w\tilde{\beta}_j\right|^2},$$

so that

$$e_{m+1}\left(\frac{1}{\pi_m(\cdot)(v-\cdot)}\right) \leqslant \frac{2|w|^{m+1}}{\left|\sqrt{v^2-1}\right|(1-|w|^2)}\prod_{j=1}^{m}\frac{\left|1+\tilde{\beta}_j^2\right|}{\left|1-w\tilde{\beta}_j\right|^2}.$$

Note that we here obtain the same result as in [19], just before the end of the proof of [19, Thm. 1]. The statement now follows from the last step in that proof. ■

The main advantage of the upper bound given in the previous theorem, is that the integral (6.5) is always computable by the Residue Theorem; see [19, Lem. 2]. We can also obtain an error bound for the rational interpolatory quadrature formulas on the unit circle, considered in the previous section. In this respect, we have the following theorem, which is a generalization of [20, Thm. 1].

THEOREM 28. *Suppose* $\mathring{\sigma}$ *is a complex measure on* $\mathbb{T}$. *Let* $\mathring{f}$ *be an analytic function on a neighborhood of the unit circle* $\mathbb{T}$, *and suppose* $I_n^{\mathring{\sigma}}(\mathring{f}) = \sum_{k=1}^{n}\mathring{\lambda}_k^{\mathring{\sigma}}\mathring{f}(z_k)$ *is a quadrature rule for the integral* $I_{\mathring{\sigma}}(\mathring{f})$ *that is exact in the space* $\mathring{\tilde{\mathcal{L}}}_p^c \cdot \mathring{\tilde{\mathcal{L}}}_{q*}$, *with* $\mathring{\tilde{\mathcal{L}}}_k := \mathring{\mathcal{L}}\{\tilde{\beta}_1,\ldots,\tilde{\beta}_k\}$ *and* $p, q < n$. *Denote the error on the nth approximation by* $\mathring{E}_n^{\mathring{\sigma}}(\mathring{f}) := I_{\mathring{\sigma}}(f) - I_n^{\mathring{\sigma}}(\mathring{f})$. *Define* $\mathring{F}_{p,q}$ *by*

$$\mathring{F}_{p,q}(z) = \mathring{\pi}_p^c(z)\mathring{\pi}_{q*}(z)\mathring{f}(z), \quad \textit{where} \quad \mathring{\pi}_m(z) := \prod_{j=1}^{m}(1-\overline{\tilde{\beta}_j}z),$$

and let $\mathring{U}$ *be the domain of analyticity of* $\mathring{F}_{p,q}$. *Further, let* $r < 1 < R$ *be such that* $\mathcal{C}_r \cup \mathcal{C}_R \subset \mathring{U}$, *where the circles* $\mathcal{C}_\rho$ *are defined as before in* (6.1). *Then*

$$\begin{aligned}\left|\mathring{E}_n(\mathring{f})\right| \leqslant \left\|\mathring{F}_{p,q}\right\|_{\mathcal{C}_r\cup\mathcal{C}_R}&\left(\int_{-\pi}^{\pi}|d\mathring{\sigma}(\theta)| + \sum_{k=1}^{n}\left|\mathring{\lambda}_k^{\mathring{\sigma}}\right|\right)\times\\ &\left(\frac{r^{q+1}}{1-r^2}\sqrt{\mathring{G}_p(r)\mathring{G}_q(r)} + \frac{R^{1-p}}{R^2-1}\sqrt{\mathring{G}_p\left(\frac{1}{R}\right)\mathring{G}_q\left(\frac{1}{R}\right)}\right),\end{aligned} \tag{6.6}$$

where $\left\|\mathring{F}_{p,q}\right\|_{\mathcal{C}_r\cup\mathcal{C}_R} := \max\left\{\left|\mathring{F}_{p,q}(z)\right| : z\in\mathcal{C}_r\cup\mathcal{C}_R\right\}$, *and*

$$\mathring{G}_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{j=1}^{m}\frac{1}{\left|1-\rho\tilde{\beta}_j e^{\mathrm{i}t}\right|^2}dt, \quad |\rho|<1,$$

with the product in the integrand equal to 1 *when* $m=0$.

Proof. The proof is exactly the same as the one of [20, Thm. 1] when replacing 'π_q' and 'ω_q' (i.e.; those with index 'q') with 'π_q^c' and 'ω_q^c' respectively. ∎

Note that for $\mathring{f}(z) = (f\circ J)(z)$, and σ and $\mathring{\sigma}$ related by (2.6), it holds that

$$J_\sigma(f) = \frac{1}{2}I_{\mathring{\sigma}}(\mathring{f}) \quad \text{and} \quad \int_{-1}^{1}|d\sigma(x)| = \frac{1}{2}\int_{-\pi}^{\pi}|d\mathring{\sigma}(\theta)|,$$

while it follows from Theorem 16 that

$$J_{n(s)}^{\sigma}(f) = \frac{1}{2}I_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \frac{1}{2}\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f}),$$

so that

$$E_{n(s)}^{\sigma}(f) = \frac{1}{2}\mathring{E}_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \frac{1}{2}\tilde{\mathring{E}}_{N(s)}^{\mathring{\sigma}}(\mathring{f}). \tag{6.7}$$

Furthermore, whenever either $s\in\{1,2\}$, or $s=0$ and condition (5.7) is satisfied, it also holds that

$$\sum_{k=1}^{n(s)}|\lambda_k^{\sigma}| = \frac{1}{2}\sum_{k=1}^{N(s)}\left|\mathring{\lambda}_k^{\mathring{\sigma}}\right| = \frac{1}{2}\sum_{k=1}^{N(s)}\left|\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\right|,$$

while we can deduce from Theorems 16–17 that the quadrature rules $I_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ have the same domain of validity; they are exact for every $\mathring{f}\in\breve{\mathring{\mathcal{S}}}_{2(n-s \bmod 2)}^{[c]} = \breve{\mathring{\mathcal{S}}}_{2(n-s \bmod 2)}^{[*]}$ (note that $2(n-s \bmod 2) = N(s)-1$ for $s=1$, but that $2(n-s \bmod 2) = N(s)$ for $s\in\{0,2\}$). As a result, the error bound (6.4) could also be obtained with the aid of (6.6) by setting $p=q=m$ and $r=1/R=\rho$, so that

$$\left(\frac{r^{q+1}}{1-r^2}\sqrt{\mathring{G}_p(r)\mathring{G}_q(r)} + \frac{R^{1-p}}{R^2-1}\sqrt{\mathring{G}_p\left(\frac{1}{R}\right)\mathring{G}_q\left(\frac{1}{R}\right)}\right) = \frac{2\rho^{m+1}}{1-\rho^2}\mathring{G}_m(\rho),$$

and noticing that for $\tilde{\beta}_j = J^{inv}(\tilde{\alpha}_j)$,

$$\mathring{G}_m(\rho) = G_m(\rho) \quad \text{and} \quad \left\|\mathring{F}_{m,m}\right\|_{\mathcal{C}_\rho\cup\mathcal{C}_{\frac{1}{\rho}}} = |\Upsilon_m|\,\|F_m\|_{\mathcal{E}_\rho}.$$

If, however, condition (5.7) is not satisfied for $s=0$, we need to consider the auxiliary quadrature rule

$$\hat{I}_{N(0)}^{\mathring{\sigma}}(\mathring{f}) = \frac{1}{2}\left[I_{N(0)}^{\mathring{\sigma}}(\mathring{f}) + \tilde{I}_{N(0)}^{\mathring{\sigma}}(\mathring{f})\right],$$

with weights

$$\hat{\mathring{\lambda}}_k^{\mathring{\sigma}} = \frac{\mathring{\lambda}_k^{\mathring{\sigma}} + \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}}{2} = \frac{\mathring{\lambda}_k^{\mathring{\sigma}} + \mathring{\lambda}_{n+k}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} + \tilde{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}}{2} = \frac{\mathring{\lambda}_{n+k}^{\mathring{\sigma}} + \tilde{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}}{2} = \hat{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}, \quad k = 1, \ldots, n,$$

where the second and fourth equality follows from the fact that the measure $\mathring{\sigma}$ is symmetric. This way, we deduce from Theorem 16 that

$$J_{n(0)}^{\sigma}(f) = \frac{1}{2}\hat{I}_{N(0)}^{\mathring{\sigma}}(\mathring{f}), \quad \sum_{k=1}^{n(0)} |\lambda_k^{\sigma}| = \frac{1}{2}\sum_{k=1}^{N(0)} \left|\hat{\mathring{\lambda}}_k^{\mathring{\sigma}}\right|, \quad \text{and} \quad E_{n(s)}^{\sigma}(f) = \frac{1}{2}\hat{\mathring{E}}_{N(s)}^{\mathring{\sigma}}(\mathring{f}).$$

Taking into account that the auxiliary quadrature rule $\hat{I}_{N(0)}^{\mathring{\sigma}}(\mathring{f})$ has domain of validity $\breve{\mathring{\mathcal{S}}}_{2n-2}^{[c]} = \breve{\mathring{\mathcal{S}}}_{2n-2}^{[*]}$, the error bound (6.4) can again be obtained with the aid of (6.6) by setting $m = p = q$, $\rho = r = 1/R$, and $\tilde{\beta}_j = J^{inv}(\tilde{\alpha}_j)$.

To conclude this section, we will give an estimate for the rate of convergence of the sequence of rational interpolatory quadrature formulas considered in the previous section. Let us first consider the case of the unit circle. For this, we need to know something about the distribution of the complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$. Let $\mathring{\nu}_n^{\beta}$ be the normalized counting measure which assigns a point mass to β_j, taking the multiplicity of β_j into account. We then have the following generalization of [9, Thm. 5.2].

Theorem 29. *Suppose $\mathring{\mu}$ is a positive measure on $\mathbb{T}$ that satisfies the Szegő condition $\int_{-\pi}^{\pi} \log \mathring{\mu}'(\theta) d\theta > -\infty$, where $\mathring{\mu}'$ is the Radon-Nikodym derivative of the measure $\mathring{\mu}$ with respect to the Lebesgue measure on $\mathbb{T}$, and let $\mathring{\sigma}$ be a complex measure on $\mathbb{T}$, such that $\mathring{\sigma} \ll \mathring{\mu}$ and*

$$\int_{-\pi}^{\pi} |\mathring{g}(\theta)|^2 \, d\mathring{\mu}(\theta) < \infty, \quad \mathring{g}(\theta) = \frac{d\mathring{\sigma}(\theta)}{d\mathring{\mu}(\theta)}.$$

Let the sequence $\mathcal{B}$ be contained in a compact subset of $\mathbb{D}$, and assume that $\mathring{\nu}_n^{\beta}$ converges to some measure $\mathring{\nu}^{\beta}$ in weak star topology. Consider the rational interpolatory quadratures $I_n^{\mathring{\sigma}}(\mathring{f})$, $n = 1, 2, \ldots$, based on the sets of n $\mathring{\mu}$-nodes and weights such that $I_n^{\mathring{\sigma}}(\mathring{f}) = I_{\mathring{\sigma}}(\mathring{f})$ for all $\mathring{f} \in \mathring{\mathcal{L}}_{p(n-1)}^{c} \cdot \mathring{\mathcal{L}}_{q(n-1)*}$, with $p(n-1) + q(n-1) = n-1$, and $\lim_{n\to\infty} q(n)/n = r \in (0,1)$. Suppose that $\mathring{f}$ is analytic in a closed and connected region G for which $\mathbb{T} \subset G$, $G \cap (\mathcal{B} \cup \mathcal{B}_*^c \cup \{0, \infty\}) = \emptyset$, where $\mathcal{B}_*^c = \{1/\beta_1, 1/\beta_2, \ldots\}$, and such that the boundary ∂G is a finite union of Jordan curves. Further, let $\mathring{E}_n^{\mathring{\sigma}}(\mathring{f}) := I_{\sigma}(f) - I_n^{\mathring{\sigma}}(\mathring{f})$ denote the error on the nth approximation. Then it holds that

$$\limsup_{n\to\infty} \left|\mathring{E}_n^{\mathring{\sigma}}(\mathring{f})\right|^{1/n} \leqslant \gamma < 1,$$

where

$$\gamma = \max\left\{\max_{z\in\partial G\cap\mathbb{D}}\{\exp[r\lambda(z)]\}, \max_{z\in\partial G\cap\mathbb{E}}\{\exp[(1-r)\lambda(1/z)]\}\right\},$$

with $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ and

$$\lambda(z) = \int \log|\zeta_z(u)| \, d\mathring{\nu}^{\beta}(u), \quad \zeta_z(u) = \frac{u - z}{1 - \overline{z}u}.$$

Proof. The proof is exactly the same as the one of [9, Thm. 5.2] when replacing '$\pi_{q(n)}$' and '$\omega_{q(n)}$' (i.e.; those with index '$q(n)$') in the proof of [9, Thm. 5.1] with '$\pi^c_{q(n)}$' and '$\omega^c_{q(n)}$' respectively. This way it holds for the para-orthogonal rational function '$\tilde{\chi}_n(z)$' in [9, Eq. (5.8)] that (see also [9, Thm. 2.4])

$$\limsup_{n\to\infty} |\tilde{\chi}_n(z)|^{1/n} = \exp\{r\lambda(z) + (1-r)\lambda(\overline{z})\}, \quad \forall z \in \mathbb{E},$$

where it is easily verified that $\lambda(\overline{z}) = -\lambda(1/z)$. ∎

Finally, we can prove the following estimate for the rate of convergence for the case of the interval.

THEOREM 30. *Suppose μ is a positive measure on I that satisfies the Szegő condition $\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty$, where μ' is the Radon-Nikodym derivative of the measure μ with respect to the Lebesgue measure on I, and let $\mathring{\sigma}$ be a complex measure on I, such that $\sigma \ll \mu$ and $\|g\|_{\mu,2} < \infty$, where g is defined as before in (5.9). Let the sequence $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$ be bounded away from I. Consider the rational interpolatory quadratures $J^{\sigma}_{n(s)}(f)$, $n = 1, 2, \ldots$, based on the sets of $n(s)$ (μ, s, α)-nodes and weights such that $J^{\sigma}_{n(s)}(f) = J_{\sigma}(f)$ for all $f \in \breve{\mathcal{L}}_{n(s)-1}$. Suppose that f is analytic in a closed connected region H for which $I \subset H$, $H \cap (\mathcal{A} \cup \{\infty\}) = \emptyset$, and such that the boundary ∂H is a finite union of Jordan curves. Further, let $E^{\sigma}_{n(s)}(f) := J_{\sigma}(f) - J^{\sigma}_{n(s)}(f)$ denote the error on the nth approximation. Then it holds that*

$$\limsup_{n(s)\to\infty} \left| E^{\sigma}_{n(s)}(f) \right|^{1/n(s)} \leqslant \kappa < 1,$$

where

$$\kappa = \max_{x\in\partial H} \left\{ \exp[\lambda(J^{inv}(x))] \right\},$$

and $\lambda(z)$, with $x = J(z)$, is defined as above in Theorem 29, with $\beta_k = J^{inv}(\alpha_k)$ for every $k > 0$.

Proof. From (6.7) it follows that

$$\begin{aligned}\limsup_{n(s)\to\infty} \left| E^{\sigma}_{n(s)}(f) \right|^{1/n(s)} &= \limsup_{n(s)\to\infty} \left(\frac{1}{2}\right)^{1/n(s)} \left| \mathring{E}^{\mathring{\sigma}}_{N(s)}(\mathring{f}) \right|^{1/n(s)} \\ &= \limsup_{n(s)\to\infty} \left\{ \left| \mathring{E}^{\mathring{\sigma}}_{N(s)}(\mathring{f}) \right|^{1/N(s)} \right\}^{N(s)/n(s)} \leqslant \gamma^2 < 1.\end{aligned}$$

So, set $\mathring{H} := \{z \in \mathbb{C} : J(z) \in H\}$ and let $\partial\mathring{H}$ denote the boundary of $\mathring{H}$. Then it follows from Theorem 29, with $r = \frac{1}{2}$ and $\partial G = \partial\mathring{H}$, that

$$\gamma^2 = \max\left\{ \max_{z\in\partial\mathring{H}\cap\mathbb{D}} \{\exp[\lambda(z)]\}, \max_{z\in\partial\mathring{H}\cap\mathbb{E}} \{\exp[\lambda(1/z)]\} \right\} = \max_{z\in\partial\mathring{H}\cap\mathbb{D}} \{\exp[\lambda(z)]\},$$

where the last equality is because $z \in \partial\mathring{H} \cap \mathbb{E}$ iff $1/z \in \partial\mathring{H} \cap \mathbb{D}$. This concludes the proof. ∎

7. Numerical examples. In this section we will illustrate the effectiveness of the quadrature formulas introduced in Section 5. For this, we consider the Chebyshev weight function of the first kind

$$d\mu(x) = \frac{dx}{\sqrt{1-x^2}}, \quad x \in I,$$

which satisfies the Szegő condition $\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty$, and for which the corresponding measure on the unit circle is the Lebesgue measure $d\mathring{\mu}(\theta) = d\theta$. Explicit expressions for the orthonormal rational functions φ_k with respect to this measure and inner product (2.2) are given in [18, Thm. 3.2], while expressions to compute the (μ, s, α)-nodes and weights in the corresponding Gauss-type quadrature formulas can be found in [15, Sect. 4]. Further, we take

$$d\sigma(x) = \frac{dx}{(1-x^2)^{\mathbf{i}/4}}, \quad x \in I, \tag{7.1}$$

for which $\int_{-1}^{1} |d\sigma(x)| = 2$ and $\left\| \frac{d\sigma}{d\mu} \right\|_{\mu,2} = \sqrt{\frac{\pi}{2}}$. The corresponding measure on the unit circle is then given by

$$d\mathring{\sigma}(\theta) = |\sin\theta|^{1-\mathbf{i}/2}\, d\theta = \frac{\left|z^2-1\right|^{1-\mathbf{i}/2}}{2^{1-\mathbf{i}/2}} d\theta, \quad z = e^{\mathbf{i}\theta}.$$

Given a function $f_i(x)$ on I, we approximate the integral $J_\sigma(f_i)$ by means of an $n(s)$-point rational interpolatory quadrature formula $J^\sigma_{n(s)}(f_i)$, based on the set of (μ, s, α)-nodes $\mathbf{x}^{[s]}_{n(s)} \subset I$, where $s \in \{0, 1, 2\}$ and $\alpha = \infty$, and with the weights $\{\lambda^\sigma_k\}_{k=1}^{n(s)}$ as defined in Lemma 22. In the examples that follow, the computations were done with the aid of $\mathsf{MAPLE}^{®}10$, with 30 digits. Since the calculation of the projection coefficients $J_\sigma(\varphi_k)$ could take a lot of time (especially for higher degrees and/or different poles[3]), we considered the auxiliary functions (see also [24, Sect. 3])

$$f^{(\alpha_k)}(x) = \left(\frac{1-\alpha_k x}{x-\alpha_k}\right)^{m_k}, \quad m_k = \#\alpha_k \text{ in } \{\alpha_1, \ldots, \alpha_k\}.$$

to speed-up the computations. The integrals $J_\sigma(\varphi_k)$ were then computed by solving the lower-triangular system of equations

$$\begin{aligned} J_\sigma(f^{(\alpha_k)}) &= \sum_{j=0}^{k} J_\mu(f^{(\alpha_k)}\varphi_j^c) \cdot J_\sigma(\varphi_j) \\ &\Leftrightarrow J_\sigma(f^{(\alpha_k)}) - \varphi_0^2 J^\mu_{n(s)}(f^{(\alpha_k)}) \cdot J_\sigma(1) = \sum_{j=1}^{k} J^\mu_{n(s)}(f^{(\alpha_k)}\varphi_j^c) \cdot J_\sigma(\varphi_j), \\ &\qquad\qquad k = 1, \ldots, n-1, \end{aligned}$$

[3] For certain degrees or choices of poles, $\mathsf{MAPLE}^{®}10$ even completely failed to compute the integral $J_\sigma(\varphi_k)$.

TABLE 7.1
The relative error, given by (7.3), in the rational interpolatory quadrature formulas for the estimation of $J_\sigma(f_{n(s),1})$, where σ is given by (7.1) and $f_{n(s),1}$ is given by (7.2).

n	$r_{n(0),1}$	$r_{n(1),1}(+1)$	$r_{n(1),1}(-1)$	$r_{n(2),1}$
2	$9.3780e-30$	$7.2139e-30$	$7.3567e-30$	$8.1616e-30$
3	$1.5784e-29$	$1.4216e-29$	$1.6002e-29$	$4.1922e-30$
4	$1.9734e-29$	$9.3931e-30$	$1.5795e-29$	$2.0998e-29$
5	$2.6967e-29$	$1.9363e-29$	$3.4059e-29$	$2.6656e-29$
6	$2.1534e-29$	$3.5910e-29$	$5.3360e-29$	$2.1231e-29$
7	$5.7863e-29$	$1.5959e-29$	$1.5255e-29$	$3.8053e-29$

where $J^\mu_{n(s)}(\cdot)$ is the $n(s)$-point rational Gauss-type quadrature formula. In the case in which $s=2$, we also needed to compute the constant $a_{n,2}$. For this, we have that

$$\begin{aligned} J_\mu(f^{(\alpha)}\varphi_n^{[\alpha]c})\cdot J_\sigma(\varphi_n^{[\alpha]}) &= J_\sigma(f^{(\alpha)}) - \sum_{j=0}^{n-1} J_\mu(f^{(\alpha)}\varphi_j^c)\cdot J_\sigma(\varphi_j) \\ &= J_\sigma(f^{(\alpha)}) - \sum_{j=0}^{n-1} J^\mu_{n(2)}(f^{(\alpha)}\varphi_j^c)\cdot J_\sigma(\varphi_j) \end{aligned}$$

where it holds for the left hand side that

$$\begin{aligned} J_\mu(f^{(\alpha)}\varphi_n^{[\alpha]c})\cdot J_\sigma(\varphi_n^{[\alpha]}) &= \frac{J_\mu(f^{(\alpha)}\varphi_n^{[\alpha]c})}{J^\mu_{n(2)}(f^{(\alpha)}\varphi_n^{[\alpha]c})}\cdot J_\sigma(\varphi_n^{[\alpha]})\cdot J^\mu_{n(2)}(f^{(\alpha)}\varphi_n^{[\alpha]c}) \\ &= a_{n,2}\cdot J^\mu_{n(2)}(f^{(\alpha)}\varphi_n^{c\,[\alpha]}). \end{aligned}$$

EXAMPLE 31. The first function $f_{n(s),1}(x)$ to be considered is given by

$$f_{n(s),1}(x) = \frac{x^{n(s)-1}}{(x-\omega)^{n-1}}, \quad \omega\in\mathbb{C}_I, \quad n>1, \tag{7.2}$$

which has poles of order $n-1$ in ω, and one pole at infinity for the case in which $s=2$. So let $\alpha_k=\omega$, $k=1,2,\ldots$, with $\omega=\frac{3+\mathbf{i}}{4}$. Table 7.1 then gives the relative error

$$r_{n(s),1} := \left|\frac{J_\sigma(f_{n(s),1}) - J^\sigma_{n(s)}(f_{n(s),1})}{J_\sigma(f_{n(s),1})}\right| \tag{7.3}$$

for several values of n. The relative errors in Table 7.1 clearly show that the integrals are approximated exactly by the rational interpolatory quadrature formulas.

EXAMPLE 32. The second function $f_2(x)$ to be considered is given by

$$f_2(x) = \sin\left(\frac{1}{x^2+\omega^2}\right), \quad \omega\in\mathbb{R}_0. \tag{7.4}$$

This function has an essential singularity in $x=\mathbf{i}\omega$ and $x=-\mathbf{i}\omega$. For $\omega>0$ but very close to 0, this function is extremely oscillatory near these singularities. Since an essential singularity can be viewed as a pole of infinite multiplicity, this suggests

TABLE 7.2
The relative error, given by (7.5), in the rational interpolatory quadrature formulas for the estimation of $J_\sigma(f_2)$, where σ is given by (7.1) and f_2 is given by (7.4).

n	$r_{n(0),2}$	$r_{n(1),2}(+1)$	$r_{n(1),2}(-1)$	$r_{n(2),2}$
3	$4.1088e-2$	$1.1244e-2$	$1.1244e-2$	$1.9547e-2$
5	$8.5077e-4$	$2.3263e-4$	$2.3263e-4$	$4.1060e-4$
9	$9.1766e-7$	$3.3781e-7$	$3.3781e-7$	$2.5981e-7$
17	$5.0893e-13$	$2.2097e-13$	$2.2097e-13$	$6.9810e-14$
33	$5.7247e-27$	$2.6760e-27$	$2.6780e-27$	$3.8649e-28$

taking $\alpha_k = (-1)^{k+1}\mathbf{i}\omega$, $k = 1, 2, \ldots$. So, let $\omega = \frac{3}{4}$. Table 7.2 then gives the relative error

$$r_{n(s),2} := \left| \frac{J_\sigma(f_2) - J^\sigma_{n(s)}(f_2)}{J_\sigma(f_2)} \right| \tag{7.5}$$

for several values of n. With $\beta = J^{inv}(3\mathbf{i}/4) = -\mathbf{i}/2$ and $\mathring{\nu}^\beta = \frac{1}{2}(\delta_\beta + \delta_{\overline{\beta}})$ (where δ_z is the unit measure whose support is the point z) we obtain from Theorem 30 the following estimation for the rate of convergence:

$$\kappa = \max_{z\in\partial\mathring{H}\cap\mathbb{D}} \left\{ \sqrt{\left| \frac{z^2-\beta^2}{1-\overline{z}^2\beta^2} \right|} \right\} > \exp[\lambda(0)] = 0.5\ .$$

Figure 7.1 graphically shows the actual rate of convergence

$$\kappa_{n(s)} := \left| J_\sigma(f_2) - J^\sigma_{n(s)}(f_2) \right|^{1/n(s)} \tag{7.6}$$

as a function of the number of interpolation points $n(s)$ for the case in which $s = 0$. The graph suggests that $\lim_{n(0)\to\infty} \kappa_{n(0)} \approx 0.1$, which is indeed less than or equal to 0.5. However, this also suggests that for a desired accuracy that is sufficiently small, the accuracy will be reached approximately three times faster than indicated by the estimated rate of convergence.

EXAMPLE 33. The last function $f_3(x)$ to be considered is given by

$$f_3(x) = \frac{\pi x/\omega}{\sinh(\pi x/\omega)}, \quad \omega \in \mathbb{R}_0, \tag{7.7}$$

which has simple poles at the integer multiples of $\mathbf{i}\omega$; thus, let

$$\alpha_k = (-1)^k \lceil k/2 \rceil \mathbf{i}\omega, \quad k = 1, 2, \ldots\ . \tag{7.8}$$

Note that (see e.g. [1, p. 85])

$$\left| \frac{\sinh(\pi x/\omega)}{\pi x/\omega} \right| = \prod_{j=1}^{\infty} \left| 1 + \frac{x^2}{(j\omega)^2} \right|,$$

so that

$$\left\| F_{n(s)-1} \right\|_{\mathcal{E}_{\rho_{n(s)}}} = \left\| F_{n-1} \right\|_{\mathcal{E}_{\rho_{n(s)}}} = \left| F_{n-1}\left(J((-1)^{n-1}\mathbf{i}\rho_{n(s)}) \right) \right|, \quad \rho_{n(s)} \in (|\beta_n|, 1),$$

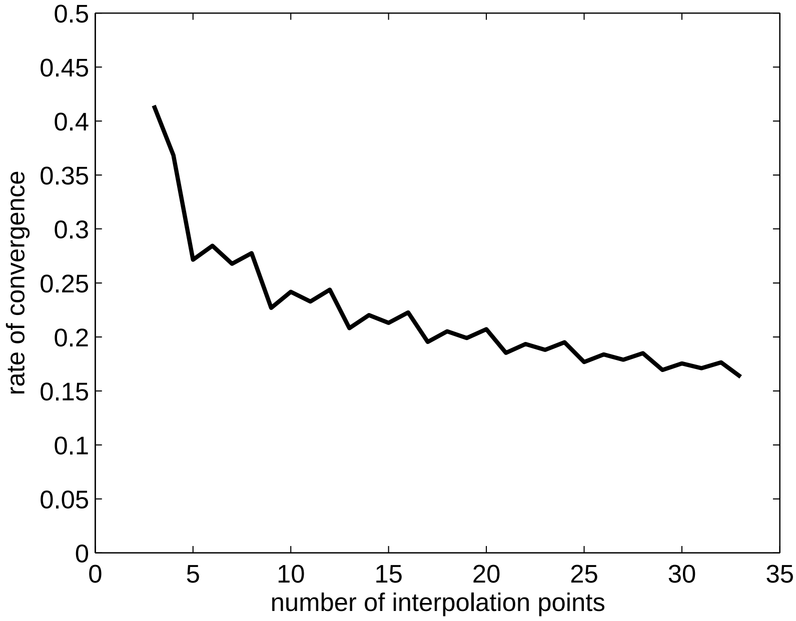

FIG. 7.1. *Rate of convergence $\kappa_{n(0)}$, given by* (7.6)*, in the rational interpolatory quadrature formula $J^{\sigma}_{n(0)}(f_2)$ for the estimation of $J_\sigma(f_2)$, where σ is given by* (7.1) *and f_2 is given by* (7.4)*.*

where $|\beta_n| = \sqrt{(\lceil n/2\rceil\omega)^2 + 1} - \lceil n/2\rceil\omega$. So, let $\omega = 1$. Tables 7.3–7.4 then give the absolute error $a_{n(s),3} := \left|J_\sigma(f_3) - J^{\sigma}_{n(s)}(f_3)\right|$ for several values of n, together with the upper bound

$$a^{[u]}_{n(s),3} := \min_{\rho_{n(s)}\in(|\beta_n|,1)} \left\{ \left(\frac{2\rho^{m_{n,s}}_{n(s)}}{1-\rho^2_{n(s)}}\right) |\Upsilon_{n-1}| \cdot \left|F_{n-1}\left(J((-1)^{n-1}\mathbf{i}\rho_{n(s)})\right)\right| \times \left(2 + \sum_{k=1}^{n(s)} |\lambda^{\sigma}_k|\right) G_{n-1}(\rho_{n(s)}) \right\}, \quad (7.9)$$

with

$$m_{n,s} = \begin{cases} 2\lceil n/2\rceil, & s = 0 \\ n, & s = 1 \\ n+1, & s = 2, \end{cases}$$

where the expression for the case in which $s = 0$ follows from the fact that $\varphi^{[\alpha]}_n(-x) \equiv (-1)^n\varphi^{[\alpha]}_n(x)$ for the given sequence of poles (7.8) and for $\alpha = \infty$, so that $J_\sigma(\varphi^{[\alpha]}_n) = 0$ whenever n is odd (see also Remark 19). Since $|J_\sigma(f_3)| \approx 1.3272$, the relative errors are of the same order.

8. Conclusion. Given a positive bounded Borel measure μ on the interval $I = [-1, 1]$, we provided convergence results in L^{μ}_2-norm to a function f of its sequence

TABLE 7.3

The absolute error $a_{n(s),3}$, $s \in \{0, 2\}$, and upper bound $a^{[u]}_{n(s),3}$, given by (7.9), in the rational interpolatory quadrature formulas for the estimation of $J_\sigma(f_3)$, where σ is given by (7.1) and f_3 is given by (7.7).

n	$a_{n(0),3}$	$a^{[u]}_{n(0),3}$	$a_{n(2),3}$	$a^{[u]}_{n(2),3}$
2	$2.6931e-1$	$1.0237e+1$	$3.2052e-2$	$5.8884e+0$
3	$1.0475e-2$	$2.8423e-1$	$4.0725e-3$	$2.8559e-1$
4	$3.5376e-3$	$4.5227e-1$	$8.1381e-5$	$1.4787e-1$
5	$7.7270e-5$	$5.0176e-3$	$3.8165e-5$	$5.0353e-3$
6	$2.0835e-5$	$8.4449e-3$	$3.9569e-7$	$1.8290e-3$
7	$2.5942e-7$	$4.5704e-5$	$9.0437e-8$	$4.5791e-5$

TABLE 7.4

The absolute error $a_{n(1),3}$ and upper bound $a^{[u]}_{n(1),3}$, given by (7.9), in the rational interpolatory quadrature formulas for the estimation of $J_\sigma(f_3)$, where σ is given by (7.1) and f_3 is given by (7.7).

n	$a_{n(1),3}(+1)$	$a^{[u]}_{n(1),3}(+1)$	$a_{n(1),3}(-1)$	$a^{[u]}_{n(1),3}(-1)$
2	$2.2220e-1$	$1.0210e+1$	$2.2071e-1$	$1.0245e+1$
3	$3.5521e-3$	$8.4133e-1$	$3.5521e-3$	$8.4133e-1$
4	$3.5242e-3$	$4.5203e-1$	$3.5207e-3$	$4.5314e-1$
5	$2.1264e-5$	$2.2622e-2$	$2.1264e-5$	$2.2622e-2$
6	$2.0992e-5$	$8.4495e-3$	$2.0986e-5$	$8.4561e-3$
7	$8.7345e-8$	$2.7964e-4$	$8.7345e-8$	$2.7964e-4$

of rational interpolating functions at the nodes of rational Gauss-type quadrature formulas associated with the measure μ. For this, we used the connection between rational Gauss-type quadrature formulas on I and certain rational Szegő quadrature formulas associated with a positive symmetric Borel measure $\mathring{\mu}$ on the complex unit circle.

As an application, we constructed rational interpolatory quadrature formulas for complex bounded measures σ on the interval and for complex bounded (not necessarily symmetric) measures $\mathring{\sigma}$ on the unit circle. Further, we gave conditions to ensure the convergence of these quadrature rules for the case of the interval (conditions for the case of the unit circle are easily obtained in a similar way).

Finally, an upper bound for the error on the nth approximation and an estimate for the rate of convergence have been provided for these quadrature rules on the interval as well as on the unit circle.

REFERENCES

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed, Dover, New York, 1972.

[2] A. Bultheel, R. Cruz-Barroso, and M. Van Barel. “On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line”, *Calcolo* 47(1):28–41, 2010.

[3] A. Bultheel, L. Daruis, and P. González-Vera. “A connection between quadrature formulas on the unit circle and the interval $[-1, 1]$”, *Journal of Computational and Applied Mathematics* 132(1):1–14, 2001.

[4] A. Bultheel, L. Daruis, and P. González-Vera. “Positive interpolatory quadrature formulas and para-orthogonal polynomials”, *Journal of Computational and Applied Mathematics* 179(1–2):97–119, 2005.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Computation of rational Szegő-Lobatto quadrature formulas", 2009. (Submitted)
[6] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Interpolation by rational functions with nodes on the unit circle", *Acta Applicandae Mathematicae* 61(1–3):101–118, 2000.
[7] A. Bultheel, P. González Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 1999.
[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Orthogonal rational functions and interpolatory product rules on the unit circle. II. Quadrature and convergence", *Analysis* 18:185–200, 1998.
[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Orthogonal rational functions and interpolatory product rules on the unit circle. III. Convergence of general sequences", *Analysis* 20:99–120, 2000.
[10] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Quadrature formulas on the unit circle based on rational functions", *Journal of Computational and Applied Mathematics* 50(1–3):159–170, 1994.
[11] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad". "Rational quadrature formulas on the unit circle with prescribed nodes and maximal domain of validity", *IMA Journal of Numerical Analysis* 2008. (Published online on August 21, 2009)
[12] R. Cruz-Barroso, P. González-Vera, and F. Perdomo-Pío. "Orthogonality, interpolation and quadratures on the unit circle and the interval $[-1, 1]$", *Journal of Computational and Applied Mathematics* 2009. (Published online on December 23, 2009)
[13] P. J. Davis. *Interpolation and approximation*, Blaisdell, New York, 1963. Reprint: Dover, 1975.
[14] K. Deckers. *Orthogonal Rational Functions: Quadrature, Recurrence and Rational Krylov*, Phd thesis, Department of Computer Science, Katholieke Universiteit Leuven, Leuven, Belgium, February 2009. 270 + lxiv pages.
[15] K. Deckers, A. Bultheel, R. Cruz-Barroso, and F. Perdomo-Pío. "Positive rational interpolatory quadrature formulas on the unit circle and the interval", 2010. (Submitted)
[16] K. Deckers, A. Bultheel, and J. Van Deun. "A generalized eigenvalue problem for quasi-orthogonal rational functions", 2010. (Submitted)
[17] K. Deckers, J. Van Deun, and A. Bultheel. "An extended relation between orthogonal rational functions on the unit circle and the interval $[-1, 1]$", *Journal of Mathematical Analysis and Applications* 334(2):1260–1275, 2007.
[18] K. Deckers, J. Van Deun, and A. Bultheel. "Rational Gauss-Chebyshev quadrature formulas for complex poles outside $[-1, 1]$", *Mathematics of Computation* 77(262):967–983, 2008.
[19] B. de la Calle Ysern. "Error bounds for rational quadrature formulae of analytic functions", *Numerische Mathematik* 101(2):251–271, 2005.
[20] B. de la Calle Ysern and P. González-Vera. "Rational quadrature formulae on the unit circle with arbitrary poles", *Numerische Mathematik* 107(4):559–587, 2007.
[21] W. Rudin. *Functional Analysis*, McGraw-Hill, New York, 1991.
[22] I. H. Sloan and W. E. Smith. "Properties of interpolatory product integration rules", *SIAM Journal on Numerical Analysis* 19(2):427–442, 1982.
[23] G. Szegő. *Orthogonal Polynomials*, vol. 33 of Amer. Math. Soc. Colloq. Publ., 4th ed., American Mathematical Society, Providence, Rhode Island, 1975. First edition 1939.
[24] J. Van Deun and A. Bultheel. " A quadrature formula based on Chebyshev rational functions", *IMA Journal of Numerical Analysis* 26(4):641–656, 2006.
[25] P. Van gucht and A. Bultheel. "A relation between orthogonal rational functions on the unit circle and the interval $[-1, 1]$", *Communications in the Analytic Theory of Continued Fractions* 8:170–182, 2000.